

SEMICLASSICAL COMPLETELY INTEGRABLE SYSTEMS : LONG-TIME DYNAMICS AND OBSERVABILITY VIA TWO-MICROLOCAL WIGNER MEASURES

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ABSTRACT. We look at the long-time behaviour of solutions to a semi-classical Schrödinger equation on the torus. We consider time scales which go to infinity when the semi-classical parameter goes to zero and we associate with each time-scale the set of semi-classical measures associated with all possible choices of initial data. On each classical invariant torus, the structure of semi-classical measures is described in terms of two-microlocal measures, obeying explicit propagation laws.

We apply this construction in two directions. We first analyse the regularity of semi-classical measures, and we emphasize the existence of a threshold : for time-scales below this threshold, the set of semi-classical measures contains measures which are singular with respect to Lebesgue measure in the “position” variable, while at (and beyond) the threshold, all the semi-classical measures are absolutely continuous in the “position” variable, reflecting the dispersive properties of the equation. Second, the techniques of two-microlocal analysis introduced in the paper are used to prove semiclassical observability estimates. The results apply as well to general quantum completely integrable systems.

1. INTRODUCTION

1.1. The Schrödinger equation in the large time and high frequency régime.

This article is concerned with the dynamics of the linear equation

$$(1) \quad \begin{cases} ih\partial_t \psi_h(t, x) = (H(hD_x) + h^2 \mathbf{V}_h(t)) \psi_h(t, x), & (t, x) \in \mathbb{R} \times \mathbb{T}^d, \\ \psi_h|_{t=0} = u_h, \end{cases}$$

on the torus $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$, with H a smooth, real-valued function on $(\mathbb{R}^d)^*$ (the dual of \mathbb{R}^d), and $h > 0$. In other words, H is a function on the cotangent bundle $T^*\mathbb{T}^d = \mathbb{T}^d \times (\mathbb{R}^d)^*$ that does not depend on the d first variables, and thus gives rise to a completely integrable Hamiltonian flow. For the sake of simplicity, we shall assume that $H \in \mathcal{C}^\infty(\mathbb{R}^d)$. However the smoothness assumption on H can be relaxed to \mathcal{C}^k , where k large enough, in most results of this article. The lower order term $\mathbf{V}_h(t)$ is a bounded self-adjoint operator (possibly depending on t and h). We assume that the map $t \mapsto \|\mathbf{V}_h(t)\|_{\mathcal{L}(L^2(\mathbb{T}^d))}$ is in $L^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, uniformly with respect to h . This condition ensures the existence of a semi-group associated with the operator $H(hD_x) + h^2 \mathbf{V}_h(t)$ (see Appendice B in [15], Proposition B.3.6).

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We are interested in the simultaneous limits $h \rightarrow 0^+$ (high frequency limit) and $t \rightarrow +\infty$ (large time evolution). Our results give a description of the limits of sequences of “position densities” $|\psi_h(t_h, x)|^2$ at times t_h that tend to infinity as $h \rightarrow 0^+$.

Remark 1.1. *In future applications, it will be interesting to note as of now that we may allow $H = H_h$ to depend on the parameter h , in such a way that H_h converges to some limit H_0 in the C^k topology on compact sets, for k sufficiently large. For instance, we allow $H_h(\xi) = H(\xi + h\omega)$, where $\omega \in (\mathbb{R}^d)^*$ is a fixed vector.*

To be more specific, let us denote by $S_h(t, s)$ the semigroup associated with the operator $H(hD_x) + h^2\mathbf{V}_h(t)$ and set $S_h^t = S_h(t, 0)$. Fix a *time scale*, that is, a function

$$\begin{aligned} \tau : \mathbb{R}_+^* &\longrightarrow \mathbb{R}_+^* \\ h &\longmapsto \tau_h, \end{aligned}$$

such that $\liminf_{h \rightarrow 0^+} \tau_h > 0$ (actually, we shall be mainly concerned in scales that go to $+\infty$ as $h \rightarrow 0^+$). Consider a family of initial conditions (u_h) , normalised in $L^2(\mathbb{T}^d)$: $\|u_h\|_{L^2(\mathbb{T}^d)} = 1$ for $h > 0$, and h -oscillating in the terminology of [20, 22], *i.e.*:

$$(2) \quad \limsup_{h \rightarrow 0^+} \left\| \mathbf{1}_{[R, +\infty[} (-h^2 \Delta) u_h \right\|_{L^2(\mathbb{T}^d)} \xrightarrow{R \rightarrow \infty} 0,$$

where $\mathbf{1}_{[R, +\infty[}$ is the characteristic function of the interval $[R, +\infty[$. Our main object of interest is the density $|S_h^t u_h|^2$, and we introduce the probability measures on \mathbb{T}^d :

$$\nu_h(t, dx) := |S_h^t u_h(x)|^2 dx;$$

the unitary character of S_h^t implies that $\nu_h \in \mathcal{C}(\mathbb{R}; \mathcal{P}(\mathbb{T}^d))$ (in what follows, $\mathcal{P}(X)$ stands for the set of probability measures on a Polish space X).

To study the long-time behavior of the dynamics, we rescale time by τ_h and look at the time-scaled probability densities:

$$(3) \quad \nu_h(\tau_h t, dx).$$

When $t \neq 0$ is fixed and τ_h grows too rapidly, it is a notoriously difficult problem to obtain a description of the limit points (in the weak-* topology) of these probability measures as $h \rightarrow 0^+$, for rich enough families of initial data u_h . See for instance [40, 38] in the case where the underlying classical dynamics is chaotic, the u_h are a family of lagrangian states, and $\tau_h = h^{-2+\epsilon}$. In completely integrable situations, such as the one we consider here, the problem is of a different nature, but rapidly leads to intricate number theoretical issues [33, 32, 34].

We soften the problem by considering the family of probability measures (3) as elements of $L^\infty(\mathbb{R}; \mathcal{P}(\mathbb{T}^d))$. Our goal will be to give a precise description of the set $\mathcal{M}(\tau)$ of their accumulation points in the weak-* topology for $L^\infty(\mathbb{R}; \mathcal{P}(\mathbb{T}^d))$, obtained as (u_h) varies among all possible sequences of initial data h -oscillating and normalised in $L^2(\mathbb{T}^d)$.

The compactness of \mathbb{T}^d ensures that $\mathcal{M}(\tau)$ is non-empty. Having $\nu \in \mathcal{M}(\tau)$ is equivalent to the existence of a sequence (h_n) going to 0 and of a normalised, h_n -oscillating sequence

(u_{h_n}) in $L^2(\mathbb{T}^d)$ such that:

$$(4) \quad \lim_{n \rightarrow +\infty} \frac{1}{\tau_{h_n}} \int_{\tau_{h_n} a}^{\tau_{h_n} b} \int_{\mathbb{T}^d} \chi(x) |S_{h_n}^t u_{h_n}(x)|^2 dx dt = \int_a^b \int_{\mathbb{T}^d} \chi(x) \nu(t, dx) dt,$$

for every real numbers $a < b$ and every $\chi \in \mathcal{C}(\mathbb{T}^d)$. In other words, we are averaging the densities $|S_h^t u_h(x)|^2$ over time intervals of size τ_h . This averaging, as we shall see, makes the study more tractable.

If case (4) occurs, we shall say that ν is obtained through the sequence (u_{h_n}) . To simplify the notation, when no confusion can arise, we shall simply write that $h \rightarrow 0^+$ to mean that we are considering a discrete sequence h_n going to 0^+ , and we shall denote by (u_h) (instead of (u_{h_n})) the corresponding family of functions.

Remark 1.2. *When the function τ is bounded, the convergence of $\nu_h(\tau_h t, \cdot)$ to an accumulation point $\nu(t, \cdot)$ is locally uniform in t . According to Egorov's theorem (see, for instance, [46]), ν can be completely described in terms of semiclassical defect measures of the corresponding sequence of initial data (u_h) , transported by the classical Hamiltonian flow $\phi_s : T^*\mathbb{T}^d \rightarrow T^*\mathbb{T}^d$ generated by H , which in this case is completely integrable :*

$$(5) \quad \phi_s(x, \xi) := (x + s dH(\xi), \xi).$$

As an example, take $\tau_h = 1$ and consider the case where the initial data u_h are coherent states : fix $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\|\rho\|_{L^2(\mathbb{R}^d)} = 1$, fix $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$, and let $u_h(x)$ be the $2\pi\mathbb{Z}^d$ -periodization of the following coherent state:

$$\frac{1}{h^{d/4}} \rho\left(\frac{x - x_0}{\sqrt{h}}\right) e^{i\frac{\xi_0}{h} \cdot x}.$$

Then $\nu_h(t, \cdot)$ converges, for every $t \in \mathbb{R}$, to:

$$\delta_{x_0 + t dH(\xi_0)}(x).$$

When the time scale τ_h is unbounded, the t -dependence of elements $\nu \in \mathcal{M}(\tau)$ is not described by such a simple propagation law. From now on we shall only consider the case where $\tau_h \xrightarrow{h \rightarrow 0} +\infty$.

The problem of describing the elements in $\mathcal{M}(\tau)$ for some time scale (τ_h) is related to several aspects of the dynamics of the flow S_h^t such as dispersive effects and unique continuation. In [4, 30] the reader will find a description of these issues in the case where the operator S_h^t is the semiclassical Schrödinger propagator $e^{iht\Delta}$ corresponding to the Laplacian on an arbitrary compact Riemannian manifold. In that setting, the time scale $\tau_h = 1/h$ appears in a natural way, since it transforms the semiclassical propagator into the non-scaled flow $e^{i\tau_h t \Delta} = e^{it\Delta}$. The possible accumulation points of sequences of probability densities of the form $|e^{it\Delta} u_h|^2$ depend on the nature of the dynamics of the geodesic flow. When the geodesic flow has the Anosov property (a very strong form of chaos, which holds on negatively curved manifolds), the results in [5] rule out concentration on sets of small dimensions, by proving lower bounds on the Kolmogorov-Sinai entropy of semiclassical defect measures. Even in the apparently simpler case that the geodesic flow is

completely integrable, different type of concentration phenomena may occur, depending on fine geometrical issues (compare the situation in Zoll manifolds [28] and on flat tori [29, 3]).

1.2. Semiclassical defect measures. Our results are more naturally described in terms of *Wigner distributions* and *semiclassical measures* (these are the semiclassical version of the *microlocal defect measures* [21, 41], and have also been called *microlocal lifts* in the recent literature about quantum unique ergodicity, see for instance the celebrated paper [27]). The *Wigner distribution* associated to u_h (at scale h) is a distribution on the cotangent bundle $T^*\mathbb{T}^d$, defined by

$$(6) \quad \int_{T^*\mathbb{T}^d} a(x, \xi) w_{u_h}^h(dx, d\xi) = \langle u_h, \text{Op}_h(a) u_h \rangle_{L^2(\mathbb{T}^d)}, \quad \text{for all } a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d),$$

where $\text{Op}_h(a)$ is the operator on $L^2(\mathbb{T}^d)$ associated to a by the Weyl quantization. The reader not familiar with these objects can consult the appendix of this article or the book [46]. For the moment, just recall that $w_{u_h}^h$ extends naturally to smooth functions χ on $T^*\mathbb{T}^d = \mathbb{T}^d \times (\mathbb{R}^d)^*$ that depend only on the first coordinate, and in this case we have

$$(7) \quad \int_{T^*\mathbb{T}^d} \chi(x) w_{u_h}^h(dx, d\xi) = \int_{\mathbb{T}^d} \chi(x) |u_h(x)|^2 dx.$$

The main object of our study will be the (time-scaled) Wigner distributions corresponding to solutions to (1):

$$w_h(t, \cdot) := w_{S_h^{\tau_h t} u_h}^h$$

The map $t \mapsto w_h(t, \cdot)$ belongs to $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$, and is uniformly bounded in that space as $h \rightarrow 0^+$ whenever (u_h) is normalised in $L^2(\mathbb{T}^d)$. Thus, one can extract subsequences that converge in the weak-* topology on $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$. In other words, after possibly extracting a subsequence, we have

$$\int_{\mathbb{R}} \int_{T^*\mathbb{T}^d} \varphi(t) a(x, \xi) w_h(t, dx, d\xi) dt \xrightarrow{h \rightarrow 0} \int_{\mathbb{R}} \int_{T^*\mathbb{T}^d} \varphi(t) a(x, \xi) \mu(t, dx, d\xi) dt$$

for all $\varphi \in L^1(\mathbb{R})$ and $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$, and the limit μ belongs to $L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$ (here $\mathcal{M}_+(X)$ denotes the set of positive Radon measures on a Polish space X).

The set of limit points thus obtained, as (u_h) varies among normalised sequences, will be denoted by $\widetilde{\mathcal{M}}(\tau)$. We shall refer to its elements as (time-dependent) semiclassical measures.

Moreover, if (u_h) is h -oscillating (see (2)), it follows that $\mu \in L^\infty(\mathbb{R}; \mathcal{P}(T^*\mathbb{T}^d))$ and identity (7) is also verified in the limit :

$$\int_a^b \int_{\mathbb{T}^d} \chi(x) |S_h^{\tau_h t} u_h(x)|^2 dx dt \xrightarrow{h \rightarrow 0} \int_a^b \int_{T^*\mathbb{T}^d} \chi(x) \mu(t, dx, d\xi) dt,$$

for every $a < b$ and every $\chi \in \mathcal{C}^\infty(\mathbb{T}^d)$. Therefore, $\mathcal{M}(\tau)$ coincides with the set of projections onto x of semiclassical measures in $\widetilde{\mathcal{M}}(\tau)$ corresponding to h -oscillating sequences [20, 22].

It is also shown in the appendix that the elements of $\widetilde{\mathcal{M}}(\tau)$ are measures that are *H-invariant*, by which we mean that they are invariant under the action of the hamiltonian flow ϕ_s defined in (5).

1.3. Results on the regularity of semiclassical measures. The main results in this article are aimed at obtaining a precise description of the elements in $\widetilde{\mathcal{M}}(\tau)$ (and, as a consequence, of those of $\mathcal{M}(\tau)$). We first present a regularity result which emphasises the critical character of the time scale $\tau_h = 1/h$ in situations in which the Hessian of H is non-degenerate, definite (positive or negative).

Theorem 1.3. (1) *If $\tau_h \ll 1/h$ then $\mathcal{M}(\tau)$ contains elements that are singular with respect to the Lebesgue measure $dtdx$. Actually, $\widetilde{\mathcal{M}}(\tau)$ contains all measures invariant by the flow ϕ_s defined in (5).*

(2) *Suppose $\tau_h \sim 1/h$ or $\tau_h \gg 1/h$. Assume that the Hessian $d^2H(\xi)$ is definite for all ξ . Then*

$$\mathcal{M}(\tau) \subseteq L^\infty(\mathbb{R}; L^1(\mathbb{T}^d)),$$

in other words the elements of $\mathcal{M}(\tau)$ are absolutely continuous with respect to $dtdx$.

The proof of (1) in Theorem 1.3 relies on the construction of examples, while the proof of (2) is based on the forthcoming Theorem 1.10, which contains a careful analysis of the case $\tau_h = 1/h$ (see section 1.4). A comparison argument between different time-scales allows to treat the case $\tau_h \gg 1/h$ (see section 1.5).

Note also that the construction leading to Theorem 1.3 (2) also yields observability results : see section 7 below. Finally, we point out in Section 1.7 that Theorem 1.3 extends to general quantum completely integrable systems. An interesting and immediate by-product of Theorem 1.3 is the following corollary.

Corollary 1.4. *Theorem 1.3(2) applies in particular when the data (u_h) are eigenfunctions of $H(hD_x)$, and shows (assuming the Hessian of H is definite) that the weak limits of the probability measures $|u_h(x)|^2 dx$ are absolutely continuous.*

Note that statement (2) of Theorem 1.3 has already been proved in the case $H(\xi) = |\xi|^2$ in [8] and [3] with different proofs (the proof in the second reference extends to the x -dependent Hamiltonian $|\xi|^2 + h^2 V(x)$). However, the extension to more general H of the method in [3] is not straightforward, even in the case where $H(\xi) = \xi \cdot A\xi$, where A is a symmetric linear map : $(\mathbb{R}^d)^* \rightarrow \mathbb{R}^d$ (i.e. the Hessian of H is constant), the difficulty arising when A has irrational coefficients.

Let us now comment on the assumptions of the theorem. We first want to emphasize that the conclusion of Theorem 1.3(2) may fail if the condition on the Hessian of H is not satisfied.

Example 1.5. *Fix $\omega \in \mathbb{R}^d$ and take $H(\xi) = \xi \cdot \omega$ and $\mathbf{V}_h(t) = 0$. Let μ_0 be an accumulation point in $\mathcal{D}'(T^*\mathbb{T}^d)$ of the Wigner distributions $(w_{u_h}^h)$ defined in (6), associated to the initial data (u_h) . Let $\mu \in \widetilde{\mathcal{M}}(\tau)$ be the limit of $w_{S_h^{\tau_h t} u_h}^h$ in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$. Then an application*

of Egorov's theorem (actually, a particularly simple adaptation of the proof of Theorem 4 in [28]) gives the relation, valid for any time scale (τ_h) :

$$\int_{T^*\mathbb{T}^d} a(x, \xi) \mu(t, dx, d\xi) = \int_{T^*\mathbb{T}^d} \langle a \rangle(x, \xi) \mu_0(dx, d\xi),$$

for any $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ and a.e. $t \in \mathbb{R}$. Here $\langle a \rangle$ stands for the average of a along the Hamiltonian flow ϕ_s , that is in our case

$$\langle a \rangle(x, \xi) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a(x + s\omega, \xi) ds.$$

Hence, as soon as ω is resonant (in the sense of §2.1) and $\mu_0 = \delta_{x_0} \otimes \delta_{\xi_0}$ for some $(x_0, \xi_0) \in T^*\mathbb{T}^d$, the measure μ will be singular with respect to $dt dx$.

It is also easy to provide counter-examples where the Hessian of H is non-degenerate, but not definite.

Example 1.6. On the two-dimensional torus \mathbb{T}^2 , consider $H(\xi) = \xi_1^2 - \xi_2^2$, where $\xi = (\xi_1, \xi_2)$. Take for $(u_h(x_1, x_2))$ the periodization of

$$\frac{1}{(2\pi h)^{1/2}} \rho\left(\frac{x_1 - x_2}{h}\right)$$

where $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfies $\|\rho\|_{L^2(\mathbb{R})} = 1$. Then the functions u_h are eigenfunctions of $H(hD_x)$ for the eigenvalue 0 and the measures $|u_h(x_1, x_2)|^2 dx_1 dx_2$ obviously concentrate on the diagonal $\{x_1 = x_2\}$.

Note however that in this example the system is *isoenergetically degenerate* at $\xi = 0$. Recall the definition of *isoenergetic non-degeneracy* : the Hamiltonian H is isoenergetically non-degenerate at ξ if for all $\eta \in (\mathbb{R}^d)^*$, and $\lambda \in \mathbb{R}$,

$$dH(\xi) \cdot \eta = 0 \text{ and } d^2H(\xi) \cdot \eta = \lambda dH(\xi) \implies (\eta, \lambda) = (0, 0).$$

Definiteness of the Hessian implies isoenergetic non-degeneracy at all ξ such that $dH(\xi) \neq 0$. In view of the previous example, one may wonder whether isoenergetic non-degeneracy is a sufficient assumption for our results. In Section 4.5 we give a sufficient set of assumptions for our results which is weaker than definiteness, but is not implied by isoenergetic non-degeneracy except in dimension $d = 2$. As a conclusion, isoenergetic non-degeneracy is sufficient for all our results in dimension $d = 2$, but not in dimensions $d \geq 3$, as is finally shown by the following counter-example :

Example 1.7. Take $d = 3$. On $(\mathbb{R}^3)^*$ consider $H(\xi) = \xi_1^2 + \xi_2^2 - \xi_3^3$, and let $u_h(x_1, x_2, x_3)$ be the periodization of

$$\frac{1}{(2\pi\epsilon)^{1/2}} \rho\left(\frac{x_2 + x_3}{\epsilon}\right) e^{i\frac{\alpha x_1 + x_2 + x_3}{h}},$$

where $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$, $\|\rho\|_{L^2(\mathbb{R})} = 1$, and $\epsilon = \epsilon(h)$ tends to 0 with $\epsilon(h) \gg h$. Note that u_h is an eigenfunction of $H(hD_x)$. The Wigner measures of (u_h) concentrate on the set

$\{\xi_1 = \alpha, \xi_2 = \xi_3 = 1\}$ where the system is isoenergetically non-degenerate if $\alpha \neq 0$. Its projection on \mathbb{T}^3 is supported on the hyperplane $\{x_2 + x_3 = 0\}$.

In Section 5 we present an example communicated to us by J. Wunsch showing that absolute continuity of the elements of $\mathcal{M}(1/h)$ may fail in the presence of a subprincipal symbol of order h^β with $\beta \in (0, 2)$ even in the case $H(\xi) = |\xi|^2$. We also show in Section 5 that absolute continuity may fail for the elements of $\mathcal{M}(1/h)$ when $H(\xi) = |\xi|^{2k}$, $k \in \mathbb{N}$ and $k > 1$; a situation where the Hessian is degenerate at $\xi = 0$.

We point out that Theorem 1.3(2) admits a microlocal refinement, which allows us to deal with more general Hamiltonians H whose Hessian is not necessarily definite at every $\xi \in \mathbb{R}^d$. Given $\mu \in \widetilde{\mathcal{M}}(\tau)$ we shall denote by $\bar{\mu}$ the image of μ under the map $\pi_2 : (x, \xi) \mapsto \xi$. For $\mathbf{V}_h(t) = \text{Op}_h(V(t, x, \xi))$ with $V \in C^\infty(\mathbb{R} \times T^*\mathbb{T}^d)$, it is shown in the appendix that $\bar{\mu}$ does not depend on t if $\tau_h \ll h^{-2}$: in this case we have $\bar{\mu} = (\pi_2)_* \mu_0$, where the measure μ_0 is an accumulation point in $\mathcal{D}'(T^*\mathbb{T}^d)$ of the sequence $(w_{u_h}^h)$. For simplicity we restrict our attention to that case in the following theorem :

Theorem 1.8. *Assume that $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ with $V \in C^\infty(\mathbb{R} \times T^*\mathbb{T}^d)$ bounded.*

Let $\mu \in \widetilde{\mathcal{M}}(1/h)$ and denote by $\mu_\xi(t, \cdot)$ the disintegration of $\mu(t, \cdot)$ with respect to the variable ξ , i.e. for every $\theta \in L^1(\mathbb{R})$ and every bounded measurable function f :

$$\int_{\mathbb{R}} \theta(t) \int_{\mathbb{T}^d \times \mathbb{R}^d} f(x, \xi) \mu(t, dx, d\xi) dt = \int_{\mathbb{R}} \theta(t) \int_{\mathbb{R}^d} \left(\int_{\mathbb{T}^d} f(x, \xi) \mu_\xi(t, dx) \right) \bar{\mu}(d\xi) dt.$$

Then for $\bar{\mu}$ -almost every ξ where $d^2H(\xi)$ is definite, the measure $\mu_\xi(t, \cdot)$ is absolutely continuous.

Let us introduce the closed set

$$C_H := \{\xi \in \mathbb{R}^d : d^2H(\xi) \text{ is not definite}\}.$$

The following consequence of Theorem 1.8 provides a refinement on Theorem 1.3(2), in which the global hypothesis on the Hessian of H is replaced by a hypothesis on the sequence of initial data.

Corollary 1.9. *Suppose $\nu \in \mathcal{M}(1/h)$ is obtained through an h -oscillating sequence (u_h) having a semiclassical measure μ_0 such that $\mu_0(\mathbb{T}^d \times C_H) = 0$. Then ν is absolutely continuous with respect to $dt dx$.*

1.4. Second-microlocal structure of the semiclassical measures. Theorem 1.8 is a consequence of a more detailed result on the structure of the elements of $\widetilde{\mathcal{M}}(1/h)$ on which we focus in this paragraph. We follow here the strategy of [3] that we adapt to a general Hamiltonian $H(\xi)$. The proof relies on a decomposition of the measure associated with the primitive submodules of $(\mathbb{Z}^d)^*$. Before stating it, we must introduce some notation.

Recall that $(\mathbb{R}^d)^*$ is the dual of \mathbb{R}^d . Later in the paper, we will sometimes identify both by working in the canonical basis of \mathbb{R}^d . We will denote by $(\mathbb{Z}^d)^*$ the lattice in $(\mathbb{R}^d)^*$ defined by $(\mathbb{Z}^d)^* = \{\xi \in (\mathbb{R}^d)^*, \xi \cdot n \in \mathbb{Z}, \forall n \in \mathbb{Z}^d\}$. We call a submodule $\Lambda \subset (\mathbb{Z}^d)^*$ primitive if

$\langle \Lambda \rangle \cap (\mathbb{Z}^d)^* = \Lambda$ (here $\langle \Lambda \rangle$ denotes the linear subspace of $(\mathbb{R}^d)^*$ spanned by Λ). Given such a submodule we define:

$$(8) \quad I_\Lambda := \left\{ \xi \in (\mathbb{R}^d)^* : dH(\xi) \cdot k = 0, \forall k \in \Lambda \right\}.$$

We note that $I_\Lambda \setminus C_H$ is a smooth submanifold.

We define also $L^p(\mathbb{T}^d, \Lambda)$ for $p \in [1, \infty]$ to be the subspace of $L^p(\mathbb{T}^d)$ consisting of the functions u such that $\widehat{u}(k) = 0$ if $k \in (\mathbb{Z}^d)^* \setminus \Lambda$ (here $\widehat{u}(k)$ stand for the Fourier coefficients of u). Given $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ and $\xi \in \mathbb{R}^d$, denote by $\langle a \rangle_\Lambda(\cdot, \xi)$ the orthogonal projection of $a(\cdot, \xi)$ on $L^2(\mathbb{T}^d, \Lambda)$:

$$(9) \quad \langle a \rangle_\Lambda(x, \xi) = \sum_{k \in \Lambda} \widehat{a}_k(\xi) \frac{e^{ikx}}{(2\pi)^d}$$

Note that if a only has frequencies in Λ , then $\langle a \rangle_\Lambda = a$.

For ω in the torus $\langle \Lambda \rangle / \Lambda$, we denote by $L_\omega^2(\mathbb{R}^d, \Lambda)$ the subspace of $L_{\text{loc}}^2(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ formed by the functions whose Fourier transform is supported in $\Lambda - \omega$. Each $L_\omega^2(\mathbb{R}^d, \Lambda)$ has a natural Hilbert space structure.

We denote by $m_{\langle a \rangle_\Lambda}(\xi)$ the operator acting on each $L_\omega^2(\mathbb{R}^d, \Lambda)$ by multiplication by $\langle a \rangle_\Lambda(\cdot, \xi)$.

Theorem 1.10. (1) Let $\mu \in \widetilde{\mathcal{M}}(1/h)$. For every primitive submodule $\Lambda \subset (\mathbb{Z}^d)^*$ there exists a positive measure $\mu_\Lambda^{\text{final}} \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$ supported on $\mathbb{T}^d \times I_\Lambda$ and invariant by the Hamiltonian flow ϕ_s such that : for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ that vanishes on $\mathbb{T}^d \times C_H$ and every $\theta \in L^1(\mathbb{R})$:

$$(10) \quad \int_{\mathbb{R}} \theta(t) \int_{T^*\mathbb{T}^d} a(x, \xi) \mu(t, dx, d\xi) dt = \sum_{\Lambda \subseteq \mathbb{Z}^d} \int_{\mathbb{R}} \theta(t) \int_{\mathbb{T}^d \times I_\Lambda} a(x, \xi) \mu_\Lambda^{\text{final}}(t, dx, d\xi) dt,$$

the sum being taken over all primitive submodules of $(\mathbb{Z}^d)^*$.

In addition, there exists a measure $\bar{\mu}_\Lambda(t)$ on $(\langle \Lambda \rangle / \Lambda) \times I_\Lambda$ and a measurable family $\{N_\Lambda(t, \omega, \xi)\}_{t \in \mathbb{R}, \omega \in \langle \Lambda \rangle / \Lambda, \xi \in I_\Lambda}$ of non-negative, symmetric, trace-class operators acting on $L_\omega^2(\mathbb{R}^d, \Lambda)$, such that the following holds:

$$(11) \quad \int_{\mathbb{T}^d \times I_\Lambda} a(x, \xi) \mu_\Lambda^{\text{final}}(t, dx, d\xi) = \int_{(\langle \Lambda \rangle / \Lambda) \times I_\Lambda} \text{Tr}(m_{\langle a \rangle_\Lambda}(\xi) N_\Lambda(t, \omega, \xi)) \bar{\mu}_\Lambda(t, d\omega, d\xi).$$

(2) If $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ with $V \in \mathcal{C}^\infty(\mathbb{R} \times T^*\mathbb{T}^d)$, then $\bar{\mu}_\Lambda$ does not depend on t , and $N_\Lambda(t, \omega, \xi)$ depends continuously on t , and solves the Heisenberg equation labelled below as $(\text{Heis}_{\Lambda, \omega, \xi})$.

When the Hessian of H is definite, formula (10) holds for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ and therefore completely describes μ .

Remark 1.11. The arguments in Section 6.1 of [3] show that Theorem 1.8 is a consequence of Theorem 1.10. Therefore, in this article only the proof of Theorem 1.10 will be presented.

Theorem 1.10 has been proved for $H(\xi) = |\xi|^2$ in [29] for $d = 2$ and in [3] for the x -dependent Hamiltonian $|\xi|^2 + h^2 V(x)$ in arbitrary dimension (in these papers the parameter ω does not appear, and all measures have Dirac masses at $\omega = 0$).

The measures $\mu_\Lambda^{\text{final}}$ in equation (10) are obtained as the final step of an iterative procedure that involves a process of successive microlocalizations along nested sequences of submanifolds in frequency space.

Theorem 1.10(2) allows to describe the dependence of μ on the parameter t . This is a subtle issue since, as was noticed in [28, 29], the semiclassical measures of the sequence of initial data (u_h) do not determine uniquely the time dependent semiclassical measure μ . Thus, when $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ with $V \in \mathcal{C}^\infty(\mathbb{R} \times T^*\mathbb{T}^d)$, the measure $\mu_\Lambda^{\text{final}}(t, dx, d\xi)$ is fully determined by the measures $\bar{\mu}_\Lambda$ and the family of operators $N_\Lambda(0, \omega, \xi)$, which are objects determined by the initial data (u_h) . The $N_\Lambda(t, \omega, \xi)$ are obtained from $N_\Lambda(0, \omega, \xi)$ by propagation along a Heisenberg equation ($\text{Heis}_{\Lambda, \omega, \xi}$), written in Theorem 3.2, which is the evolution equation of operators that comes from the following Schrödinger equation in $L_\omega^2(\mathbb{R}^d, \Lambda)$:

$$(S_{\Lambda, \omega, \xi}) \quad i\partial_t v = \left(\frac{1}{2} d^2 H(\xi) D_y \cdot D_y + \langle V(\cdot, \xi) \rangle_\Lambda \right) v.$$

This process gives an explicit construction of μ in terms of the initial data. Full details on the structure of these objects are provided in Sections 3 and 4.

Theorem 1.10 is stated for the time scale $\tau_h = 1/h$; if $\tau_h \ll 1/h$, the elements of $\widetilde{\mathcal{M}}(\tau)$ can also be described by a similar result (see Section 4.3) involving expression (10). However, in that case, the propagation law involves classical transport rather than propagation along a Schrödinger flow, and as a result Theorem 1.3(2) does not hold for $\tau_h \ll 1/h$.

Second microlocalisation has been used in the 80's for studying propagation of singularities (see [6, 7, 14, 26]). The two-microlocal construction performed here is in the spirit of that done in [37, 17, 18] in Euclidean space in the context of semi-classical measures. We also refer the reader to the articles [43, 44, 45] for related work regarding the study of the wave-front set of solutions to semiclassical integrable systems.

When the Hessian of H is constant Theorem 1.10 gives a complement to the results announced in [3] (where the argument was only valid when the Hessian has rational coefficients).

1.5. Hierarchy of time scales. In this section, we discuss the dependence of the set $\mathcal{M}(\tau)$ on the time scale τ . The following proposition allows to derive Theorem 1.3(2) for $\tau_h \gg 1/h$ from the result about $\tau_h = 1/h$. Denote by $\mathcal{M}_{\text{av}}(\tau)$ the subset of $\mathcal{P}(\mathbb{T}^d)$ consisting of measures of the form:

$$\int_0^1 \nu(t, \cdot) dt, \quad \text{where } \nu \in \text{Conv } \mathcal{M}(\tau).$$

where $\text{Conv } X$ stands for the convex hull of a set $X \subset L^\infty(\mathbb{R}; \mathcal{P}(\mathbb{T}^d))$ with respect to the weak-* topology. We have the following result.

Proposition 1.12. *Suppose (τ_h) and (τ'_h) are time scales tending to infinity and such that $\tau'_h \ll \tau_h$. Then:*

$$\mathcal{M}(\tau) \subseteq L^\infty(\mathbb{R}; \mathcal{M}_{\text{av}}(\tau')).$$

It is also important to clarify the link between the time-dependent Wigner distributions and those associated with eigenfunctions. Eigenfunctions are the most commonly studied objects in the field of quantum chaos, however, we shall see that they do not necessarily give full information about the time-dependent Wigner distributions. For the sake of simplicity, we state the results that follow in the case $\mathbf{V}_h(t) = 0$, although they easily generalise to the case in which $\mathbf{V}_h(t)$ does not depend on t . Start noting that the spectrum of $H(hD_x)$ coincides with $H(h\mathbb{Z}^d)$; given $E_h \in \text{sp}(H(hD_x))$ the corresponding normalised eigenfunctions are of the form:

$$(12) \quad u_h(x) = \sum_{H(hk)=E_h} c_k^h e^{ik \cdot x}, \quad \text{with} \quad \sum_{k \in \mathbb{Z}^d} |c_k^h|^2 = \frac{1}{(2\pi)^d}.$$

In addition, one has:

$$\nu_h(\tau_h t, \cdot) = |S_h^{\tau_h t} u_h|^2 = |u_h|^2,$$

independently of (τ_h) and t . Let us denote by $\mathcal{M}(\infty)$ the set of accumulation points in $\mathcal{P}(\mathbb{T}^d)$ of sequences $|u_h|^2$ where (u_h) varies among all possible h -oscillating sequences of normalised eigenfunctions (12), we have

$$\mathcal{M}(\infty) \subseteq \mathcal{M}(\tau).$$

As a consequence of Theorem 1.3, we obtain the following result.

Corollary 1.13. *All eigenfunction limits $\mathcal{M}(\infty)$ are absolutely continuous under the definiteness assumption on the Hessian of H .*

A time scale of special importance is the one related to the minimal spacing of eigenvalues : define

$$(13) \quad \tau_h^H := h \sup \left\{ |E_h^1 - E_h^2|^{-1} : E_h^1 \neq E_h^2, E_h^1, E_h^2 \in H(h\mathbb{Z}^d) \right\}.$$

It is possible to have $\tau_h^H = \infty$: for instance, if $H(\xi) = |\xi|^\alpha$ with $0 < \alpha < 1$ or $H(\xi) = \xi \cdot A\xi$ with A a real symmetric matrix that is not proportional to a matrix with rational entries (this is the content of the Oppenheim conjecture, settled by Margulis [13, 31]). In some other situations, such as $H(\xi) = |\xi|^\alpha$ with $\alpha > 1$, (13) is finite : $\tau_h^H = h^{1-\alpha}$.

Proposition 1.14. *If $\tau_h \gg \tau_h^H$ one has:*

$$\mathcal{M}(\tau) = \text{Conv } \mathcal{M}(\infty).$$

This result is a consequence of the more general results presented in Section 6.

Note that Proposition 1.14 allows to complete the description of $\mathcal{M}(\tau)$ in the case $H(\xi) = |\xi|^2$ as the time scale varies.

Remark 1.15. Suppose $H(\xi) = |\xi|^2$, or more generally, that $\tau_h^H \sim 1/h$ and the Hessian of H is definite. Then:

$$\begin{aligned} & \text{if } \tau_h \ll 1/h, \quad \exists \nu \in \mathcal{M}(\tau) \text{ such that } \nu \perp dt dx; \\ & \text{if } \tau_h \sim 1/h, \quad \mathcal{M}(\tau) \subseteq L^\infty(\mathbb{R}; L^1(\mathbb{T}^d)); \\ & \text{if } \tau_h \gg 1/h \quad \mathcal{M}(\tau) = \text{Conv } \mathcal{M}(\infty). \end{aligned}$$

Finally, we point out that in this case the regularity of semiclassical measures can be precised. The elements in $\mathcal{M}(\infty)$ are trigonometric polynomials when $d = 2$, as shown in [23]; and in general they are more regular than merely absolutely continuous, see [1, 23, 36]. The same phenomenon occurs with those elements in $\mathcal{M}(1/h)$ that are obtained through sequences whose corresponding semiclassical measures do not charge $\{\xi = 0\}$, see [2].

1.6. Application to semiclassical and non-semiclassical observability estimates.

As was already shown in [3] for the case $H(\xi) = |\xi|^2$ the characterization of the structure of the elements in $\mathcal{M}(1/h)$ implies quantitative, unique continuation-type estimates for the solutions of the Schrödinger equation (1) known as observability inequalities. This is the case again in this setting; here we shall prove the following result.

Theorem 1.16. Let $U \subset \mathbb{T}^d$ open and nonempty, $T > 0$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\text{supp } \chi \cap C_H = \emptyset$. Assume that $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ with $V \in \mathcal{C}^\infty(\mathbb{R} \times T^*\mathbb{T}^d)$ bounded. Then the following are equivalent:

i) Semiclassical observability estimate. There exists $C = C(U, T, \chi) > 0$ and $h_0 > 0$ such that:

$$(14) \quad \|\chi(hD_x)u\|_{L^2(\mathbb{T}^d)}^2 \leq C \int_0^T \int_U \left| S_h^{t/h} \chi(hD_x)u(x) \right|^2 dx dt,$$

for every $u \in L^2(\mathbb{T}^d)$ and $h \in (0, h_0]$.

ii) Unique continuation in $\widetilde{\mathcal{M}}(1/h)$. For every $\mu \in \widetilde{\mathcal{M}}(1/h)$ with $\bar{\mu}(\text{supp } \chi) \neq 0$ and $\bar{\mu}(C_H) = 0$ (recall that $\bar{\mu}$ is the image of μ under the projection π_2) one has:

$$\int_0^T \mu(t, U \times \text{supp } \chi) dt \neq 0.$$

Besides, any of i) or ii) is implied by the following statement.

iii) Unique continuation for the family of Schrödinger equations $(S_{\Lambda, \omega, \xi})$. For every $\Lambda \subset \mathbb{Z}^d$, every $\xi \in \text{supp } \chi$ with $\Lambda \subseteq dH(\xi)^\perp$ and every $\omega \in \langle \Lambda \rangle / \Lambda$, one has the following unique continuation property: if $v \in \mathcal{C}(\mathbb{R}; L_\omega^2(\mathbb{R}^d, \Lambda))$ solves the Schrödinger equation $(S_{\Lambda, \omega, \xi})$ and $v|_{(0, T) \times U} = 0$ then $v = 0$.

This result will be proved as a consequence of the structure Theorem 1.10.

Remark 1.17. The unique continuation property for $(S_{\Lambda, \omega, \xi})$ stated in Theorem 1.16, iii) is known to hold in any of the following two cases:

i) $V(\cdot, \xi)$ is analytic in (t, x) for every ξ . This is a consequence of Holmgren's uniqueness theorem (see [42])

ii) $V(\cdot, \xi)$ is smooth (or even continuous outside of a set of null Lebesgue measure) for every ξ and does not depend on t (see Theorem 1.20 below).

Corollary 1.18. *Let U , T , χ , and $\mathbf{V}_h(t)$ be as in Theorem 1.16, and suppose that V satisfies any of the two conditions in Remark 1.17. Then the semiclassical observability estimate (14) holds.*

The nature of the observability estimate (14) is better appreciated when H is itself quadratic. Suppose:

$$H_{A,\theta}(\xi) = \frac{1}{2}A(\xi + \theta) \cdot (\xi + \theta),$$

where $\theta \in \mathbb{R}^d$, A is a definite real matrix, and denote by \overline{S}^t the (non-semiclassical) propagator, starting at $t = 0$, associated to $H_{A,\theta}(D_x) + V(t, \cdot)$. Clearly, the propagator $S_h^{t/h}$ associated to $H_{A,h\theta}(hD_x) + h^2V(t, \cdot)$ coincides with \overline{S}^t in this case.

Corollary 1.19. *Let $U \subset \mathbb{T}^d$ be a nonempty open set, and $T > 0$. Let $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ with $V \in C^\infty(\mathbb{R} \times T^*\mathbb{T}^d)$ bounded. Suppose that the following unique continuation result holds:*

For every $\Lambda \subset \mathbb{Z}^d$ and every $\xi \in \mathbb{R}^d$ with $\Lambda \subseteq dH(\xi)^\perp$, if $v \in \mathcal{C}(\mathbb{R}; L_\omega^2(\mathbb{R}^d, \Lambda))$, $\omega \in \langle \Lambda \rangle / \Lambda$, solves:

$$(15) \quad i\partial_t v = (H_{A,\theta}(D_y) + \langle V(\cdot, \xi) \rangle_\Lambda) v$$

and $v|_{(0,T) \times U} = 0$ then $v = 0$.

Then there exist $C > 0$ such that for every $u \in L^2(\mathbb{T}^d)$ one has:

$$(16) \quad \|u\|_{L^2(\mathbb{T}^d)}^2 \leq C \int_0^T \int_U |\overline{S}^t u(x)|^2 dx dt.$$

Note that an estimate such as (16) implies a unique continuation result for solutions to (1): $\overline{S}^t u|_U = 0$ for $t \in (0, T) \implies u = 0$. Corollary 1.19 shows in particular that this (weaker) unique continuation property for family of quadratic Hamiltonians in equations (15) actually implies the stronger estimate (16). We also want to stress the fact that Corollary 1.19 establishes the unique continuation property for perturbations of pseudodifferential type from the analogous property for perturbations that are merely multiplication by a potential.

It should be also mentioned that the proof of Theorem 4 in [3] can be adapted almost word by word to prove estimate (16) in the case when V does not depend on t , *without relying* in any *a priori* unique continuation result except those for eigenfunctions. In fact, the function V can be supposed less regular than smooth: it suffices that it is continuous outside of a set of null Lebesgue measure.

Theorem 1.20. *Suppose V only depends on x ; let $U \subset \mathbb{T}^d$ a nonempty open set, and let $T > 0$. Then (16) holds; in particular, any solution $\overline{S}^t u$ that vanishes identically on $(0, T) \times U$ must vanish everywhere.*

In the proof of Theorem 1.20, unique continuation for solutions to the time dependent Schrödinger equation is replaced by a unique continuation result for eigenfunctions of $H_{A,\theta}(D_x) + V(t, \cdot)$. This allows to reduce the proof of (16) to that of a semiclassical observability estimate (14) with a cut-off χ vanishing close to $\xi = 0$. At this point, the validity of (16) is reduced to the validity of the corresponding estimate on \mathbb{T}^{d-1} . The proof of (16) is completed by applying an argument of induction on the dimension d . We refer the reader to the proof of [3], Theorem 4 for additional details (see also [30] for a proof in a simpler case in $d = 2$).

Let us finally mention that Theorem 1.20 was first proved in the case $H(\xi) = |\xi|^2$ in [24] for $V = 0$, in [10] for $d = 2$ and in [3] for general d , the three results having rather different proofs. We also refer the reader to [4, 9, 25, 30] for additional results and references concerning observability inequalities in the context of Schrödinger-type equations.

1.7. Generalization to quantum completely integrable systems. Our results may be transferred to more general completely integrable systems as follows. Let (M, dx) be a compact manifold of dimension d , equipped with a density dx . Assume we have a family $(\hat{A}_1, \dots, \hat{A}_d)$ of d commuting self-adjoint h -pseudodifferential operators of order 0 in h . By this, we mean an operator $a(x, hD_x)$ where a is in some classical symbol class S^l , or may even have an asymptotic expansion $a \sim \sum_{k=0}^{+\infty} h^k a_k$ in this S^l (the term a_0 will then be called the principal symbol). Let $\tilde{H} = f(\hat{A}_1, \dots, \hat{A}_d)$ where $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth. Let $A = (A_1, \dots, A_d) : T^*M \rightarrow \mathbb{R}^d$ be the principal symbols of the operators \hat{A}_i . Note that the commutation $[\hat{A}_i, \hat{A}_j] = 0$ implies the Poisson commutation $\{A_i, A_j\} = 0$. Assume that there is an open subset W of \mathbb{R}^d and a symplectomorphism $T : \mathbb{T}^d \times W \rightarrow A^{-1}(W)$ with $A_i \circ T = \xi_j$ (note that, by Arnold-Liouville Theorem, this situation occurs locally where the differentials of the A_i are linearly independent). Then, there exists a Fourier integral operator $\hat{U} : L^2(\mathbb{T}^d) \rightarrow L^2(M)$ associated with T , such that $\hat{U}\hat{U}^* = I + O(h^\infty)$ microlocally on $A^{-1}(W)$, and such that

$$\hat{U}^* \hat{A}_j \hat{U} = hD_{x_j} + \sum_{k \geq 1} h^k S_{j,k}(hD_x)$$

on $\mathbb{T}^d \times W$ with $S_{j,k} \in \mathcal{C}^\infty(\mathbb{R}^d)$ (see [12], Theorem 78 (1)).

This may be used to generalize our results to the equation

$$(17) \quad \begin{cases} ih\partial_t \psi_h(t, x) = \left(\hat{H} + h^2 V \right) \psi_h(t, x), & (t, x) \in \mathbb{R} \times M, \\ \psi_h|_{t=0} = u_h, \end{cases}$$

where V is a pseudodifferential operator of order 0.

If a is smooth, compactly supported inside $A^{-1}(W)$, if χ is supported in $A^{-1}(W)$ taking the value 1 on the support of a , and if t stays in a compact set of \mathbb{R} , we have

$$\text{Op}_h(a) S^{\tau_h t} u_h = \text{Op}_h(a) S^{\tau_h t} \text{Op}_h(\chi) u_h + o(1)$$

as long as $\tau_h \ll h^{-2}$ (or for all τ_h if $V = 0$) and

$$\text{Op}_h(a)S^{\tau_h t}\text{Op}_h(\chi)u_h = \text{Op}_h(a)\hat{U}\hat{U}^*S^{\tau_h t}\hat{U}\hat{U}^*\text{Op}_h(\chi)u_h + O(h^\infty).$$

Note that $\hat{U}^*S^{\tau_h t}\hat{U}\hat{U}^*\text{Op}_h(\chi)u_h$ coincides modulo $o(1)$ with $\tilde{S}^{\tau_h t}\hat{U}^*\text{Op}_h(\chi)u_h$ where $\tilde{S}^{\tau_h t}$ is the propagator associated to

$$(18) \quad ih\partial_t\psi_h(t, x) = \left(f_h(hD_x) + h^2\hat{U}^*V\hat{U}\right)\psi_h(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{T}^d.$$

where $f_h(\xi) = f(\xi + \sum_{k \geq 1} h^k S_k(\xi))$.

The semiclassical measures associated with equation (17), when restricted to $A^{-1}(W)$, are exactly the images under T of the semiclassical measures coming from (18) supported on $\mathbb{T}^d \times W$. Applying Theorem 1.3 to the solutions of (18) and transporting the statement by the symplectomorphism T , we obtain the following result :

Theorem 1.21. *If $\tau_h \geq h^{-1}$ then the semiclassical measures associated with solutions of (18) are absolutely continuous measures of the lagrangian tori $A^{-1}(\xi)$, for $\bar{\mu}$ -almost every $\xi \in V$ such that $d^2f(\xi)$ is definite.*

The observability results could also be rephrased in this more general setting.

1.8. Organisation of the paper. When $\tau_h \leq 1/h$, the key argument of this article is a second microlocalisation on primitive submodules which is the subject of Section 2 and leads to Theorems 2.5 and 2.6. Sections 3 and 4 are devoted to the proof of these two theorems. At that stage of the paper, the proofs of Theorem 1.10 and Theorem 1.3(2) when $\tau_h \sim 1/h$ are then achieved. Examples are developed in Section 5 in order to prove Theorem 1.3(1). Finally, the results concerning hierarchy of time-scales are proved in Section 6 (and lead to Theorem 1.3 for $\tau_h \gg 1/h$), whereas the proof of Theorem 1.16 is given in Section 7.

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2. TWO-MICROLOCAL ANALYSIS OF INTEGRABLE SYSTEMS ON \mathbb{T}^d

In this section, we develop the two-microlocal analysis of the elements of $\widetilde{\mathcal{M}}(\tau)$ that will be at the core of the proof of Theorems 1.3, 1.8 and 1.10 in the case where $\tau_h \leq 1/h$. From now on, we shall assume that the time scale (τ_h) satisfies:

$$(19) \quad (h\tau_h) \quad \text{is a bounded sequence.}$$

Note however that the discussion of section 2.1 does not require this assumption.

2.1. Invariant measures and a resonant partition of phase-space. As in [3], the first step in our strategy to characterise the elements in $\widetilde{\mathcal{M}}(\tau)$ consists in introducing a partition of phase-space $T^*\mathbb{T}^d$ according to the order of “resonance” of its elements, that induces a decomposition of the measures $\mu \in \widetilde{\mathcal{M}}(\tau)$.

Using the duality $((\mathbb{R}^d)^*, \mathbb{R}^d)$, we denote by $A^\perp \subset \mathbb{R}^d$ the orthogonal of a set $A \in (\mathbb{R}^d)^*$ and by $B^\perp \subset (\mathbb{R}^d)^*$ the orthogonal of a set $B \in \mathbb{R}^d$. Recall that \mathcal{L} is the family of all primitive submodules of $(\mathbb{Z}^d)^*$ and that with each $\Lambda \in \mathcal{L}$, we associate the set I_Λ defined in (8): $I_\Lambda = dH^{-1}(\Lambda^\perp)$. Denote by $\Omega_j \subset \mathbb{R}^d$, for $j = 0, \dots, d$, the set of resonant vectors of order exactly j , that is:

$$\Omega_j := \{\xi \in (\mathbb{R}^d)^* : \text{rk } \Lambda_\xi = d - j\},$$

where

$$\Lambda_\xi := \{k \in (\mathbb{Z}^d)^* : k \cdot dH(\xi) = 0\} = dH(\xi)^\perp \cap \mathbb{Z}^d.$$

Note that the sets Ω_j form a partition of $(\mathbb{R}^d)^*$, and that $\Omega_0 = dH^{-1}(\{0\})$; more generally, $\xi \in \Omega_j$ if and only if the Hamiltonian orbit $\{\phi_s(x, \xi) : s \in \mathbb{R}\}$ issued from any $x \in \mathbb{T}^d$ in the direction ξ is dense in a subtorus of \mathbb{T}^d of dimension j . The set $\Omega := \bigcup_{j=0}^{d-1} \Omega_j$ is usually called the set of *resonant* momenta, whereas $\Omega_d = (\mathbb{R}^d)^* \setminus \Omega$ is referred to as the set of *non-resonant* momenta. Finally, write

$$(20) \quad R_\Lambda := I_\Lambda \cap \Omega_{d-\text{rk } \Lambda}.$$

Saying that $\xi \in R_\Lambda$ is equivalent to any of the following statements:

- (i) for any $x_0 \in \mathbb{T}^d$ the time-average $\frac{1}{T} \int_0^T \delta_{x_0+tdH(\xi)}(x) dt$ converges weakly, as $T \rightarrow \infty$, to the Haar measure on the torus $x_0 + \mathbb{T}_{\Lambda^\perp}$. Here, we have used the notation $\mathbb{T}_{\Lambda^\perp} := \Lambda^\perp / (2\pi\mathbb{Z}^d \cap \Lambda^\perp)$, which is a torus embedded in \mathbb{T}^d ;
- (ii) $\Lambda_\xi = \Lambda$.

Moreover, if $\text{rk } \Lambda = d - 1$ then $R_\Lambda = dH^{-1}(\Lambda^\perp \setminus \{0\}) = I_\Lambda \setminus \Omega_0$. Note that,

$$(21) \quad (\mathbb{R}^d)^* = \bigsqcup_{\Lambda \in \mathcal{L}} R_\Lambda,$$

that is, the sets R_Λ form a partition of $(\mathbb{R}^d)^*$. As a consequence, any measure $\mu \in \mathcal{M}_+(T^*\mathbb{R}^d)$ decomposes as

$$(22) \quad \mu = \sum_{\Lambda \in \mathcal{L}} \mu|_{\mathbb{T}^d \times R_\Lambda}.$$

Therefore, the analysis of a measure μ reduces to that of $\mu|_{\mathbb{T}^d \times R_\Lambda}$ for all primitive submodule Λ . Given an H -invariant measure μ , it turns out that $\mu|_{\mathbb{T}^d \times R_\Lambda}$ are utterly determined by the Fourier coefficients of μ in Λ . Indeed, define the complex measures on \mathbb{R}^d :

$$\widehat{\mu}_k := \int_{\mathbb{T}^d} \frac{e^{-ik \cdot x}}{(2\pi)^{d/2}} \mu(dx, \cdot), \quad k \in \mathbb{Z}^d,$$

so that, in the sense of distributions,

$$\mu(x, \xi) = \sum_{k \in \mathbb{Z}^d} \widehat{\mu}_k(\xi) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}.$$

Then, the following proposition holds.

Proposition 2.1. *Let $\mu \in \mathcal{M}_+(T^*\mathbb{T}^d)$ and $\Lambda \in \mathcal{L}$. The distribution:*

$$\langle \mu \rangle_\Lambda(x, \xi) := \sum_{k \in \Lambda} \widehat{\mu}_k(\xi) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}$$

is a finite, positive Radon measure on $T^\mathbb{T}^d$.*

Moreover, if μ is a positive H -invariant measure on $T^\mathbb{T}^d$, then every term in the decomposition (22) is a positive H -invariant measure, and*

$$(23) \quad \mu|_{\mathbb{T}^d \times R_\Lambda} = \langle \mu \rangle_\Lambda|_{\mathbb{T}^d \times R_\Lambda}.$$

Besides, identity (23) is equivalent to the fact that $\mu|_{\mathbb{T}^d \times R_\Lambda}$ is invariant by the translations

$$(x, \xi) \longmapsto (x + v, \xi), \quad \text{for every } v \in \Lambda^\perp.$$

The proof of Proposition 2.1 follows the lines of those of Lemmas 6 and 7 of [3]. We also point out that this decomposition depends on the function H through the definition of I_Λ . In the following, our aim is to determine μ restricted to $\mathbb{T}^d \times R_\Lambda$ for any $\Lambda \in \mathcal{L}$.

2.2. Second microlocalization on a resonant submanifold. Let (u_h) be a bounded sequence in $L^2(\mathbb{T}^d)$ and suppose (after extraction of a subsequence) that its Wigner distributions $w_h(t) = w_{S_h^{t\tau_h} u_h}^h$ converge to a semiclassical measure $\mu \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$ in the weak-* topology of $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$.

Given $\Lambda \in \mathcal{L}$, the purpose of this section is to study the measure $\mu|_{\mathbb{T}^d \times R_\Lambda}$ by performing a second microlocalization along I_Λ in the spirit of [17, 18, 19, 37, 35] and [3, 29]. By Proposition 2.1, it suffices to characterize the action of $\mu|_{\mathbb{T}^d \times R_\Lambda}$ on test functions having only x -Fourier modes in Λ . With this in mind, we shall introduce two auxiliary “distributions” which describe more precisely how $w_h(t)$ concentrates along $\mathbb{T}^d \times I_\Lambda$. They are actually not mere distributions, but lie in the dual of the class of symbols \mathcal{S}_Λ^1 that we define below.

In what follows, we fix $\xi_0 \in R_\Lambda$ such that $d^2H(\xi_0)$ is definite and, by applying a cut-off in frequencies to the data, we restrict our discussion to normalised sequences of initial data (u_h) that satisfy:

$$\widehat{u_h}(k) = 0, \quad \text{for } hk \in \mathbb{R}^d \setminus B(\xi_0; \epsilon/2),$$

where $B(\xi_0, \epsilon/2)$ is the ball of radius $\epsilon/2$ centered at ξ_0 . The parameter $\epsilon > 0$ is taken small enough, in order that

$$d^2H(\xi) \text{ is definite for all } \xi \in B(\xi_0, \epsilon);$$

this implies that $I_\Lambda \cap B(\xi_0, \epsilon)$ is a submanifold of dimension $d - \text{rk } \Lambda$, everywhere transverse to $\langle \Lambda \rangle$, the vector subspace of $(\mathbb{R}^d)^*$ generated by Λ . Note that this is actually achieved

under the weaker hypothesis that $d^2H(\xi)$ is non-singular and defines a definite bilinear form on $\langle \Lambda \rangle \times \langle \Lambda \rangle$ (Section 4.5 gives a set of assumptions which is weaker than definiteness but sufficient for our results). By eventually reducing ϵ , we have

$$B(\xi_0, \epsilon/2) \subset (I_\Lambda \cap B(\xi_0, \epsilon)) \oplus \langle \Lambda \rangle,$$

by which we mean that any element $\xi \in B(\xi_0, \epsilon/2)$ can be decomposed in a unique way as $\xi = \sigma + \eta$ with $\sigma \in I_\Lambda \cap B(\xi_0, \epsilon)$ and $\eta \in \langle \Lambda \rangle$. We thus get a map

$$(24) \quad \begin{aligned} F : B(\xi_0, \epsilon/2) &\longrightarrow (I_\Lambda \cap B(\xi_0, \epsilon)) \times \langle \Lambda \rangle \\ \xi &\longmapsto (\sigma(\xi), \eta(\xi)) \end{aligned}$$

With this decomposition of the space of frequencies, we associate two-microlocal test-symbols.

Definition 2.2. We denote by \mathcal{S}_Λ^1 the class of smooth functions $a(x, \xi, \eta)$ on $T^*\mathbb{T}^d \times \langle \Lambda \rangle$ that are:

- (i) compactly supported on $(x, \xi) \in T^*\mathbb{T}^d$, $\xi \in B(\xi_0, \epsilon/2)$,
- (ii) homogeneous of degree zero at infinity w.r.t. $\eta \in \langle \Lambda \rangle$, i.e. such that there exist $R_0 > 0$ and $a_{\text{hom}} \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d \times \mathbb{S}\langle \Lambda \rangle)$ with

$$a(x, \xi, \eta) = a_{\text{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right), \quad \text{for } |\eta| > R_0 \text{ and } (x, \xi) \in T^*\mathbb{T}^d$$

(we have denoted by $\mathbb{S}\langle \Lambda \rangle$ the unit sphere in $\langle \Lambda \rangle \subseteq (\mathbb{R}^d)^*$, identified later on with the sphere at infinity);

- (iii) such that their non vanishing Fourier coefficients (in the x variable) correspond to frequencies $k \in \Lambda$:

$$a(x, \xi, \eta) = \sum_{k \in \Lambda} \widehat{a}_k(\xi, \eta) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}.$$

We will also express this fact by saying that a has only x -Fourier modes in Λ .

The index 1 in the notation \mathcal{S}_Λ^1 refers to the fact that we have added *one* variable (η) to the standard class of symbols corresponding to the second microlocalisation. In Section 4, we will perform successive higher order microlocalisations corresponding to the addition of $k \geq 1$ variables and we will consider spaces denoted \mathcal{S}_Λ^k .

For $a \in \mathcal{S}_\Lambda^1$, we introduce the notation

$$\text{Op}_h^\Lambda(a(x, \xi, \eta)) := \text{Op}_h(a(x, \xi, \tau_h \eta(\xi))).$$

Notice that, for all $\beta \in \mathbb{N}^d$,

$$(25) \quad \left\| \partial_\xi^\beta (a(x, h\xi, \tau_h \eta(h\xi))) \right\|_{L^\infty} \leq C_\beta (\tau_h h)^{|\beta|}.$$

The Calderón-Vaillancourt theorem (see [11] or the appendix of [3] for a precise statement) therefore ensures that there exist $N \in \mathbb{N}$ and $C_N > 0$ such that

$$(26) \quad \forall a \in \mathcal{S}_\Lambda^1, \quad \|\mathrm{Op}_h^\Lambda(a)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_N \sum_{|\alpha| \leq N} \|\partial_{x,\xi,\eta}^\alpha a\|_{L^\infty},$$

since $(h\tau_h)$ is assumed to be bounded. Therefore the family of operators $\mathrm{Op}_h^\Lambda(a)$ is a bounded family of $L^2(\mathbb{T}^d)$.

We are going to use this formalism to decompose the Wigner transform $w_h(t)$. Let $\chi \in \mathcal{C}_c^\infty(\langle \Lambda \rangle)$ be a nonnegative cut-off function that is identically equal to one near the origin. For $a \in \mathcal{S}_\Lambda^1$, $R > 1$, $\delta < 1$, we decompose a into: $a(x, \xi, \eta) = \sum_{j=1}^3 a_j(x, \xi, \eta)$ with

$$a_1(x, \xi, \eta) := a(x, \xi, \eta) \left(1 - \chi\left(\frac{\eta}{R}\right)\right) \left(1 - \chi\left(\frac{\eta(\xi)}{\delta}\right)\right),$$

$$(27) \quad a_2(x, \xi, \eta) := a(x, \xi, \eta) \left(1 - \chi\left(\frac{\eta}{R}\right)\right) \chi\left(\frac{\eta(\xi)}{\delta}\right),$$

$$(28) \quad a_3(x, \xi, \eta) := a(x, \xi, \eta) \chi\left(\frac{\eta}{R}\right).$$

Since any smooth compactly supported function with Fourier modes in Λ can be viewed as an element of \mathcal{S}_Λ^1 (which is constant in the variable η), this induces a decomposition of the Wigner distribution:

$$w_h(t) = w_{h,R,\delta}^{I_\Lambda}(t) + w_{I_\Lambda,h,R}(t) + w_{h,R,\delta}^{I_\Lambda^c}(t),$$

where:

$$\langle w_{h,R,\delta}^{I_\Lambda}(t), a \rangle := \int_{T^*\mathbb{T}^d} a_2(x, \xi, \tau_h \eta(\xi)) w_h(t) (dx, d\xi),$$

$$(29) \quad \langle w_{I_\Lambda,h,R}(t), a \rangle := \int_{T^*\mathbb{T}^d} a_3(x, \xi, \tau_h \eta(\xi)) w_h(t) (dx, d\xi),$$

and

$$\langle w_{h,R,\delta}^{I_\Lambda^c}(t), a \rangle := \int_{T^*\mathbb{T}^d} a_1(x, \xi, \tau_h \eta(\xi)) w_h(t) (dx, d\xi),$$

that we shall analyse in the limits $h \rightarrow 0^+$, $R \rightarrow +\infty$ and $\delta \rightarrow 0$ (taken in that order).

The distributions $w_{h,R,\delta}^{I_\Lambda}(t)$ and $w_{I_\Lambda,h,R}(t)$ can be expressed for all $t \in \mathbb{R}$ by

$$(30) \quad \langle w_{h,R,\delta}^{I_\Lambda}(t), a \rangle = \langle u_h, S_h^{\tau_h t^*} \mathrm{Op}_h^\Lambda(a_2) S_h^{\tau_h t} u_h \rangle_{L^2(\mathbb{T}^d)},$$

$$(31) \quad \langle w_{I_\Lambda,h,R}^\Lambda(t), a \rangle = \langle u_h, S_h^{\tau_h t^*} \mathrm{Op}_h^\Lambda(a_3) S_h^{\tau_h t} u_h \rangle_{L^2(\mathbb{T}^d)}.$$

As a consequence of (26), both $w_{h,R,\delta}^{I_\Lambda}$ and $w_{I_\Lambda,h,R}$ are bounded in $L^\infty(\mathbb{R}; (\mathcal{S}_\Lambda^1)')$.

The first observation is that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \lim_{h \rightarrow 0} \int_{\mathbb{R}} \theta(t) \left\langle w_{h,R,\delta}^{I_\Lambda^c}(t), a \right\rangle dt \\ = \int_{\mathbb{R}} \int_{T^*\mathbb{T}^d} \theta(t) a_{\text{hom}} \left(x, \xi, \frac{\eta(\xi)}{|\eta(\xi)|} \right) \mu(t, dx, d\xi) \big|_{\mathbb{T}^d \times I_\Lambda^c} dt \end{aligned}$$

where $\mu \in \widetilde{\mathcal{M}}(\tau_h)$ is the semiclassical measure obtained through the sequence (u_h) . Since $R_\Lambda \subset I_\Lambda$, the restriction of the measure thus obtained to $\mathbb{T}^d \times R_\Lambda$ vanishes, and we do not need to further analyse the term involving the distribution $w_{h,R,\delta}^{I_\Lambda^c}(t)$.

Then, after possibly extracting subsequences, one defines limiting objects $\tilde{\mu}_\Lambda$ and $\tilde{\mu}^\Lambda$ such that for every $\varphi \in L^1(\mathbb{R})$ and $a \in \mathcal{S}_\Lambda^1$,

$$\int_{\mathbb{R}} \varphi(t) \left\langle \tilde{\mu}^\Lambda(t, \cdot), a \right\rangle dt := \lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \varphi(t) \left\langle w_{h,R,\delta}^{I_\Lambda}(t), a \right\rangle dt,$$

and

$$(32) \quad \int_{\mathbb{R}} \varphi(t) \left\langle \tilde{\mu}_\Lambda(t, \cdot), a \right\rangle dt := \lim_{R \rightarrow \infty} \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \varphi(t) \left\langle w_{I_\Lambda, h, R}(t), a \right\rangle dt.$$

From the decomposition $w_h(t) = w_{h,R,\delta}^{I_\Lambda}(t) + w_{I_\Lambda, h, R}(t) + w_{h,R,\delta}^{I_\Lambda^c}(t)$ (when testing against symbols having Fourier modes in Λ), it is immediate that the measure $\mu(t, \cdot) \big|_{\mathbb{T}^d \times R_\Lambda}$ is related to $\tilde{\mu}^\Lambda$ and $\tilde{\mu}_\Lambda$ according to the following Proposition.

Proposition 2.3. *Let*

$$\mu^\Lambda(t, \cdot) := \int_{\langle \Lambda \rangle} \tilde{\mu}^\Lambda(t, \cdot, d\eta) \big|_{\mathbb{T}^d \times R_\Lambda}, \quad \mu_\Lambda(t, \cdot) := \int_{\langle \Lambda \rangle} \tilde{\mu}_\Lambda(t, \cdot, d\eta) \big|_{\mathbb{T}^d \times R_\Lambda}.$$

Then both $\mu^\Lambda(t, \cdot)$ and $\mu_\Lambda(t, \cdot)$ are H -invariant positive measures on $T^\mathbb{T}^d$ and satisfy:*

$$(33) \quad \mu(t, \cdot) \big|_{\mathbb{T}^d \times R_\Lambda} = \mu^\Lambda(t, \cdot) + \mu_\Lambda(t, \cdot).$$

This proposition motivates the analysis of the structure of the accumulation points $\tilde{\mu}_\Lambda(t, \cdot)$ and $\tilde{\mu}^\Lambda(t, \cdot)$. It turns out that both $\tilde{\mu}^\Lambda$ and $\tilde{\mu}_\Lambda$ have some extra regularity in the variable x , although for two different reasons. Our next two results form one of the key steps towards the proof of Theorem 1.3.

Remark 2.4. *All the results in this section remain valid if the Hamiltonian H_h depends on h as stated in Remark 1.1. The proofs are completely analogous, the only difference being that the resonant manifolds I_Λ and the coordinate system $F = (\sigma, \eta)$ now also vary with h . Therefore, the definitions of $w_{h,R,\delta}^{I_\Lambda}$ and $w_{I_\Lambda, h, R}$ should be modified accordingly.*

2.3. Properties of two-microlocal semiclassical measures. We define, for $(x, \xi, \eta) \in T^*\mathbb{T}^d \times (\langle \Lambda \rangle \setminus \{0\})$ and $s \in \mathbb{R}$,

$$\begin{aligned} \phi_s^0(x, \xi, \eta) &:= (x + sdH(\xi), \xi, \eta), \\ \phi_s^1(x, \xi, \eta) &:= \left(x + sd^2H(\sigma(\xi)) \cdot \frac{\eta}{|\eta|}, \xi, \eta \right). \end{aligned}$$

This second definition extends in an obvious way to $\eta \in \mathbb{S}\langle\Lambda\rangle$ (the sphere at infinity). On the other hand, the map $(x, \xi, \eta) \mapsto \phi_{s|\eta|}^1(x, \xi, \eta)$ extends to $\eta = 0$.

We first focus on the measure $\tilde{\mu}^\Lambda$. We point out that because of the existence of $R_0 > 0$ and of $a_{\text{hom}} \in C_c^\infty(T^*\mathbb{T}^d \times \mathbb{S}\langle\Lambda\rangle)$ such that

$$a(x, \xi, \eta) = a_{\text{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right), \quad \text{for } |\eta| \geq R_0,$$

the value $\langle w_{h,R,\delta}^{I_\Lambda}(t), a \rangle$ only depends on a_{hom} . Therefore, the limiting object $\tilde{\mu}^\Lambda(t, \cdot) \in (\mathcal{S}_\Lambda^1)'$ is zero-homogeneous in the last variable $\eta \in \mathbb{R}^d$, supported at infinity, and, by construction, it is supported on $\xi \in I_\Lambda$. This can be also expressed as the fact that $\tilde{\mu}^\Lambda$ is a “distribution” on $\mathbb{T}^d \times I_\Lambda \times \overline{\langle\Lambda\rangle}$ (where $\overline{\langle\Lambda\rangle}$ is the compactification of $\langle\Lambda\rangle$ by adding the sphere $\mathbb{S}\langle\Lambda\rangle$ at infinity) supported on $\{\eta \in \mathbb{S}\langle\Lambda\rangle\}$. Moreover, we have the following result.

Theorem 2.5. *$\tilde{\mu}^\Lambda(t, \cdot)$ is a positive measure on $\mathbb{T}^d \times I_\Lambda \times \overline{\langle\Lambda\rangle}$ supported on the sphere at infinity $\mathbb{S}\langle\Lambda\rangle$ in the variable η . Besides, for a.e. $t \in \mathbb{R}$, the measure $\tilde{\mu}^\Lambda(t, \cdot)$ satisfies the invariance properties:*

$$(34) \quad (\phi_s^0)_* \tilde{\mu}^\Lambda(t, \cdot) = \tilde{\mu}^\Lambda(t, \cdot), \quad (\phi_s^1)_* \tilde{\mu}^\Lambda(t, \cdot) = \tilde{\mu}^\Lambda(t, \cdot), \quad s \in \mathbb{R}.$$

Note that this result holds whenever $\tau_h \ll 1/h$ or $\tau_h = 1/h$. This is in contrast with the situation we encounter when dealing with $\tilde{\mu}_\Lambda(t, \cdot)$. The regularity of this object indeed depends on the properties of the scale.

Theorem 2.6. (1) *The distributions $\tilde{\mu}_\Lambda(t, \cdot)$ are supported on $\mathbb{T}^d \times I_\Lambda \times \langle\Lambda\rangle$ and are continuous with respect to $t \in \mathbb{R}$.*

(2) *If $\tau_h \ll 1/h$ then $\tilde{\mu}_\Lambda(t, \cdot)$ is a positive measure.*

(3) *If $\tau_h = 1/h$, the projection of $\tilde{\mu}_\Lambda(t, \cdot)$ on $T^*\mathbb{T}^d$ is a positive measure, whose projection on \mathbb{T}^d is absolutely continuous with respect to the Lebesgue measure.*

(4) *If $\tau_h \ll 1/h$, then $\tilde{\mu}_\Lambda$ satisfy the following propagation law:*

$$(35) \quad \forall t \in \mathbb{R}, \quad \tilde{\mu}_\Lambda(t, x, \xi, \eta) = (\phi_{t|\eta|}^1)_* \tilde{\mu}_\Lambda(0, x, \xi, \eta).$$

Note that (4) implies the continuous dependence of $\tilde{\mu}_\Lambda(t, \cdot)$ with respect to t in the case $\tau_h \ll 1/h$. For $\tau_h = 1/h$ the dependence of $\tilde{\mu}_\Lambda(t, x, \xi, \eta)$ on t will be investigated in Section 3.

Remark 2.7. *Consider the decomposition $\mu(t, \cdot) = \sum_{\Lambda \in \mathcal{L}} \mu^\Lambda(t, \cdot) + \sum_{\Lambda \in \mathcal{L}} \mu_\Lambda(t, \cdot)$ given by Proposition 2.3. When $\tau_h = 1/h$, Theorem 2.6(3) implies that the second term defines a positive measure whose projection on \mathbb{T}^d is absolutely continuous with respect to the Lebesgue measure.*

Theorem 2.6 calls for a few comments.

The fact that the distribution $\tilde{\mu}_\Lambda$ is supported on $\mathbb{T}^d \times I_\Lambda \times \langle\Lambda\rangle$ is straightforward. Indeed, we have for all t ,

$$(36) \quad \langle w_{I_\Lambda, h, R}(t), a(x, \xi, \eta) \rangle = \langle w_{I_\Lambda, h, R}(t), a(x, \sigma(\xi), \eta) \rangle + O(\tau_h^{-1})$$

since, by (26),

$$\begin{aligned} \text{Op}_h^\Lambda(a_3(x, \xi, \eta)) &= \text{Op}_h^\Lambda(a(x, \sigma(\xi) + \tau_h^{-1}\eta, \eta)\chi(\eta/R)) \\ &= \text{Op}_h^\Lambda(a(x, \sigma(\xi), \eta)\chi(\eta/R)) + O(\tau_h^{-1}) \end{aligned}$$

where the $O(\tau_h^{-1})$ term is understood in the sense of the operator norm of $\mathcal{L}(L^2(\mathbb{R}^d))$ and depends on R (the fact that we first let h go to 0^+ is crucial here).

When $\tau_h \ll 1/h$ the quantization of our symbols generates a semi-classical pseudodifferential calculus with gain $h\tau_h$. The operators $\text{Op}_h^\Lambda(a)$ are semiclassical both in ξ and η . This implies that the accumulation points $\tilde{\mu}^\Lambda$ and $\tilde{\mu}_\Lambda$ are positive measures (see for instance [35] or [19]).

When $\tau_h = 1/h$, we will see in Theorem 3.2 in Section 3 that the distributions $\tilde{\mu}_\Lambda(t, \cdot)$ satisfy an invariance law that can be interpreted in terms of a Schrödinger flow type propagator.

Let us now comment on the invariance by the flows. Note first that of major importance is the observation that for all $\xi \in (\mathbb{R}^d)^* \setminus C_H$ (recall that C_H stands for the points where the Hessian $d^2H(\xi)$ is not definite) we have the decomposition $\mathbb{R}^d = \Lambda^\perp \oplus d^2H(\xi)\langle\Lambda\rangle$. Therefore, the flows ϕ_s^0 and ϕ_s^1 are independent on $\mathbb{T}^d \times (R_\Lambda \setminus C_H) \times \langle\Lambda\rangle$. Then, the following remark holds:

Remark 2.8. *In the case where $\text{rk } \Lambda = 1$ then (34) implies that, for a.e. $t \in \mathbb{R}$, and for any ν in the 1-dimensional space $\langle\Lambda\rangle$, the measure $\tilde{\mu}^\Lambda(t, \cdot)|_{\mathbb{T}^d \times R_\Lambda \times \langle\Lambda\rangle}$ is invariant under*

$$(x, \sigma, \eta) \longmapsto (x + d^2H(\sigma) \cdot \nu, \sigma, \eta).$$

On the other hand, the invariance by the Hamiltonian flow and Proposition 2.1, imply that $\tilde{\mu}^\Lambda(t, \cdot)|_{\mathbb{T}^d \times R_\Lambda \times \langle\Lambda\rangle}$ is also invariant under

$$(x, \sigma, \eta) \longmapsto (x + v, \sigma, \eta)$$

for every v in the hyperplane Λ^\perp . Using the independence of the different flows and the fact that the Hessian $d^2H(\sigma)$ is definite on the support of $\tilde{\mu}^\Lambda(t, \cdot)|_{\mathbb{T}^d \times R_\Lambda \times \langle\Lambda\rangle}$, we conclude that the measure $\tilde{\mu}^\Lambda(t, \cdot)|_{\mathbb{T}^d \times R_\Lambda \times \langle\Lambda\rangle}$ is constant in $x \in \mathbb{T}^d$ in this case. For $\text{rk } \Lambda > 1$, we will develop a similar argument thanks to successive microlocalisations (see section 4).

In the next subsection, we prove the invariance properties stated in Theorems 2.5 and 2.6 (2) and (4). For $\tau_h = h^{-1}$ the detailed analysis of the measure $\tilde{\mu}_\Lambda$ is performed in section 3 and the proof of the absolute continuity of its projection on $T^*\mathbb{T}^d$ is done in section 4.

2.4. Invariance properties of two-microlocal semiclassical measures.

Proof of Theorem 2.5. The positivity of $\tilde{\mu}^\Lambda(t, \cdot)$ can be deduced following the lines of [19] §2.1, or those of the proof of Theorem 1 in [21]; see also the appendix of [3]. The proof of invariance of $\tilde{\mu}^\Lambda(t, \cdot)$ under ϕ_s^0 is similar to the proof of invariance of μ under ϕ_s done in the appendix.

Let us now check the invariance property (34). Using (27), we have (along any convergent subsequence)

$$(37) \quad \begin{aligned} \int_{\mathbb{R}} \varphi(t) \langle \tilde{\mu}^\Lambda(t, \cdot), a \rangle dt &= \lim_{\delta \rightarrow 0} \lim_{R \rightarrow +\infty} \lim_{h \rightarrow 0} \int_{\mathbb{R}} \varphi(t) \langle w_{h,R,\delta}^{I_\Lambda}(t), a \rangle dt \\ &= \lim_{\delta \rightarrow 0} \lim_{R \rightarrow +\infty} \lim_{h \rightarrow 0} \int_{\mathbb{R}} \varphi(t) \langle w_h(t), a_2(x, \xi, \tau_h \eta(\xi)) \rangle dt. \end{aligned}$$

Notice that the symbol

$$a_2 \circ \phi_s^1(x, \xi, \eta) = a_2 \left(x + sd^2 H(\sigma(\xi)) \frac{\eta}{|\eta|}, \xi, \eta \right),$$

is a well-defined element of \mathcal{S}_Λ^1 , since, for fixed R , a_2 is identically equal to zero near $\eta = 0$; moreover

$$\forall \omega \in \mathbb{S}(\Lambda), \quad (a_2 \circ \phi_s^1)_{\text{hom}}(x, \xi, \omega) = a_{\text{hom}}(x + sd^2 H(\sigma(\xi))\omega, \xi, \omega).$$

We write

$$\frac{d}{ds} \Big|_{s=0} (a_2 \circ \phi_s^1)(x, \xi, \tau_h \eta(\xi)) = \left(d^2 H(\sigma(\xi)) \frac{\eta(\xi)}{|\eta(\xi)|} \right) \cdot \partial_x a_2(x, \xi, \tau_h \eta(\xi)).$$

Using the Taylor expansion

$$d^2 H(\sigma(\xi))\eta(\xi) + G(\xi)[\eta(\xi), \eta(\xi)] = dH(\xi) - dH(\sigma(\xi))$$

where

$$(38) \quad G(\xi) = \int_0^1 d^3 H(\sigma(\xi) + t\eta(\xi))(1-t) dt,$$

is uniformly bounded, and taking into account the fact that $\eta(\xi) = O(\delta)$ on the support of a_2 , we have

$$\left(d^2 H(\sigma(\xi)) \frac{\eta(\xi)}{|\eta(\xi)|} \right) \cdot \partial_x a_2(x, \xi, \tau_h \eta(\xi)) = \left(\frac{dH(\xi) - dH(\sigma(\xi))}{|\eta(\xi)|} \right) \cdot \partial_x a_2(x, \xi, \tau_h \eta(\xi)) + O(\delta).$$

Because a_2 has only x -Fourier coefficients in Λ and $dH(\sigma(\xi)) \in \Lambda^\perp$, we can write

$$\left(\frac{dH(\xi) - dH(\sigma(\xi))}{|\eta(\xi)|} \right) \cdot \partial_x a_2(x, \xi, \tau_h \eta(\xi)) = \left(\frac{dH(\xi)}{|\eta(\xi)|} \right) \cdot \partial_x a_2(x, \xi, \tau_h \eta(\xi)).$$

Note now that

$$\begin{aligned} \text{Op}_h \left(\frac{dH(\xi)}{|\eta(\xi)|} \cdot \partial_x a_2(x, \xi, \tau_h \eta(\xi)) \right) &= \frac{i}{h} \left[H(hD_x) + h^2 \mathbf{V}_h(t), \text{Op}_h \left(\frac{a_2(x, \xi, \tau_h \eta(\xi))}{|\eta(\xi)|} \right) \right] \\ &\quad + O(h) + O\left(\frac{h\tau_h}{R}\right). \end{aligned}$$

For the last term, we have only used that $\mathbf{V}_h(t)$ is a bounded operator on L^2 and

$$\left\| \text{Op}_h \left(\frac{a_2(x, \xi, \tau_h \eta(\xi))}{|\eta(\xi)|} \right) \right\|_{L^2 \rightarrow L^2} = O\left(\frac{\tau_h}{R}\right)$$

since $\frac{1}{|\eta(\xi)|} = O\left(\frac{\tau_h}{R}\right)$ on the support of a_2 .

To conclude, take a test function $\varphi(t) \in \mathcal{C}_c^\infty(\mathbb{R})$ (those are dense in $L^1(\mathbb{R})$).

$$\begin{aligned}
& \int_{\mathbb{R}} \varphi(t) \left\langle w_{h,R,\delta}^{I_\Lambda}(t), \frac{d}{ds} \Big|_{s=0} (a_2 \circ \phi_s^1)(x, \xi, \tau_h \eta(\xi)) \right\rangle dt \\
&= \int_{\mathbb{R}} \varphi(t) \left\langle S_h^{\tau_h t} u_h, \frac{i}{h} \left[H(hD_x) + h^2 \mathbf{V}_h(t), \text{Op}_h \left(\frac{a_2(x, \xi, \tau_h \eta(\xi))}{|\eta(\xi)|} \right) \right] S_h^{\tau_h t} u_h \right\rangle dt \\
&\quad + O(h) + O\left(\frac{h\tau_h}{R}\right) + O(\delta) \\
&= \frac{1}{\tau_h} \int_{\mathbb{R}} \varphi(t) \frac{d}{dt} \left\langle S_h^{\tau_h t} u_h, \text{Op}_h \left(\frac{a_2(x, \xi, \tau_h \eta(\xi))}{|\eta(\xi)|} \right) S_h^{\tau_h t} u_h \right\rangle dt + O(h) + O\left(\frac{h\tau_h}{R}\right) + O(\delta) \\
&= -\frac{1}{\tau_h} \int_{\mathbb{R}} \varphi'(t) \left\langle S_h^{\tau_h t} u_h, \text{Op}_h \left(\frac{a_2(x, \xi, \tau_h \eta(\xi))}{|\eta(\xi)|} \right) S_h^{\tau_h t} u_h \right\rangle dt + O(h) + O\left(\frac{h\tau_h}{R}\right) + O(\delta) \\
&\quad = O(\tau_h^{-1}) + O(h) + O\left(\frac{h\tau_h}{R}\right) + O(\delta).
\end{aligned}$$

Taking the limit $h \rightarrow 0$ followed by $R \rightarrow +\infty$ and $\delta \rightarrow 0$, we obtain

$$\int_{\mathbb{R}} \varphi(t) \left\langle \tilde{\mu}^\Lambda(t), \frac{d}{ds} \Big|_{s=0} (a \circ \phi_s^1) \right\rangle dt = 0$$

for any φ and a , which ends the proof of Theorem 2.5. \square

Proof of (1), (2) and (4) of Theorem 2.6 for $h\tau_h \rightarrow 0$. The statement on the support of the measure $\tilde{\mu}_\Lambda$ has already been discussed in Section 2.2 (after Remark 2.7). The positivity of $\tilde{\mu}_\Lambda$ is standard once we notice that the two-scale quantization admits the gain $h\tau_h$ (in view of (25)). Note also that (4) implies the continuous dependence with respect to t .

Thus let us prove part (4) of the theorem. The propagation law (and hence, the continuity with respect to t) is proved as follows. Let

$$\tilde{\phi}_t^1(x, \xi, \eta) = \phi_{t|\eta|}^1(x, \xi, \eta) = (x + td^2H(\sigma(\xi))\eta, \xi, \eta).$$

We write

$$\frac{d}{dt} \Big|_{t=0} a_3 \circ \tilde{\phi}_t^1(x, \xi, \tau_h \eta(\xi)) = \tau_h d^2H(\sigma(\xi))\eta(\xi) \cdot \partial_x a_3(x, \xi, \tau_h \eta(\xi))$$

and the same argument as in the previous proof now yields

$$\tau_h d^2H(\sigma(\xi))\eta(\xi) \cdot \partial_x a_3(x, \xi, \tau_h \eta(\xi)) = \tau_h dH(\xi) \cdot \partial_x a_3(x, \xi, \tau_h \eta(\xi)) + O\left(\frac{R^2}{\tau_h}\right)$$

where we now use that $|\eta(\xi)| = O\left(\frac{R}{\tau_h}\right)$ on the support of a_3 . Note now that

$$\tau_h \text{Op}_h(dH(\xi) \cdot \partial_x a_3(x, \xi, \tau_h \eta(\xi))) = \frac{i\tau_h}{h} [H(hD_x) + h^2 \mathbf{V}_h(t), \text{Op}_h(a_3(x, \xi, \tau_h \eta(\xi)))] + O(h\tau_h),$$

using only the fact that $\mathbf{V}_h(t)$ is a bounded operator.

To conclude, take a test function $\varphi(t) \in C_c^\infty(\mathbb{R})$.

$$\begin{aligned}
& \int_{\mathbb{R}} \varphi(t) \left\langle w_{I_\Lambda, h, R}(t), \frac{d}{dt} \Big|_{t=0} a_3 \circ \tilde{\phi}_t^1(x, \xi, \tau_h \eta(\xi)) \right\rangle dt \\
&= \int_{\mathbb{R}} \varphi(t) \left\langle S_h^{\tau_h t} u_h, \frac{i\tau_h}{h} [H(hD_x) + h^2 \mathbf{V}_h(t), \text{Op}_h(a_3(x, \xi, \tau_h \eta(\xi)))] S_h^{\tau_h t} u_h \right\rangle dt \\
&\quad + O(h\tau_h) + O\left(\frac{R^2}{\tau_h}\right) \\
&= \int_{\mathbb{R}} \varphi(t) \frac{d}{dt} \langle S_h^{\tau_h t} u_h, \text{Op}_h(a_3(x, \xi, \tau_h \eta(\xi))) S_h^{\tau_h t} u_h \rangle dt + O(h\tau_h) + O\left(\frac{R^2}{\tau_h}\right) \\
&= - \int_{\mathbb{R}} \varphi'(t) \langle S_h^{\tau_h t} u_h, \text{Op}_h(a_3(x, \xi, \tau_h \eta(\xi))) S_h^{\tau_h t} u_h \rangle dt + O(h\tau_h) + O\left(\frac{R^2}{\tau_h}\right)
\end{aligned}$$

Taking the limit $h \rightarrow 0$ followed by $R \rightarrow +\infty$, we obtain

$$\int_{\mathbb{R}} \varphi(t) \left\langle \tilde{\mu}_\Lambda(t), \frac{d}{dt} \Big|_{t=0} (a \circ \tilde{\phi}_t^1) \right\rangle dt = - \int_{\mathbb{R}} \varphi'(t) \langle \tilde{\mu}_\Lambda(t), a \rangle dt$$

for any φ and a , which is the announced result. \square

3. REGULARITY AND TRANSPORT OF $\tilde{\mu}_\Lambda$.

In this section, we suppose $\tau_h = 1/h$ and we prove statement (3) of Theorem 2.6. This constitutes a first step towards the proof of Theorem 1.10 (and Theorem 1.3(2)) which will be achieved in Section 4. In Theorem 3.2 below, we give a description of the measure $\tilde{\mu}_\Lambda$. The first part of our result implies in particular that the projection of $\tilde{\mu}_\Lambda$ onto \mathbb{T}^d is absolutely continuous. For this result to hold we only assume that $\mathbf{V}_h(t)$ is a general bounded self-adjoint perturbation as described in the Introduction. The second part of Theorem 3.2 shows that $\tilde{\mu}_\Lambda$ satisfies a propagation law that involves a Heisenberg equation. For that part we need to assume that $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ for some smooth bounded symbol V .

Recall that for ω in the torus $\langle \Lambda \rangle / \Lambda$, we denote by $L_\omega^2(\mathbb{R}^d, \Lambda)$ the subspace of $L_{\text{loc}}^2(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ formed by the functions whose Fourier transform is supported in $\Lambda - \omega$. In other words, $f \in L_\omega^2(\mathbb{R}^d, \Lambda)$ if and only if $f \in L_{\text{loc}}^2(\mathbb{R}^d)$ and:

$$\begin{aligned}
& f(\cdot + v) = f, \quad \forall v \in \Lambda^\perp, \\
(39) \quad & f(\cdot + k) = e^{-i\omega \cdot k} f, \quad \forall k \in 2\pi\mathbb{Z}^d,
\end{aligned}$$

where, recall, Λ^\perp stands for the orthogonal of Λ in the duality sense. Clearly, $f \in L_\omega^2(\mathbb{R}^d, \Lambda)$ if and only if there exists $g \in L^2(\mathbb{T}^d, \Lambda)$ (the set of L^2 function on \mathbb{T}^d which have Fourier modes in Λ) such that $f(x) = e^{-i\omega \cdot x} g(x)$; this characterization induces a natural Hilbert structure on $L_\omega^2(\mathbb{R}^d, \Lambda)$.

We introduce an auxiliary lattice $\tilde{\Lambda} \subset 2\pi\mathbb{Z}^d$ such that $2\pi\mathbb{Z}^d \subset \Lambda^\perp \oplus \tilde{\Lambda}$. Let $D_{\tilde{\Lambda}} \subset \langle \tilde{\Lambda} \rangle$ be a fundamental domain of the action of $\tilde{\Lambda}$ on $\langle \tilde{\Lambda} \rangle$. Each space $L_\omega^2(\mathbb{R}^d, \Lambda)$ is naturally

isomorphic to $L^2(D_{\tilde{\Lambda}})$, (simply by extending by continuity the restriction of functions in $C^\infty(\mathbb{R}^d) \cap L_\omega^2(\mathbb{R}^d, \Lambda)$). Under this isomorphism, the norm in $L^2(D_{\tilde{\Lambda}})$ equals a factor $|D_{\tilde{\Lambda}}|^{1/2} / (2\pi)^{d/2}$ times the norm in $L_\omega^2(\mathbb{R}^d, \Lambda)$.

Introduce a vector bundle \mathfrak{F} over $(\langle \Lambda \rangle / \Lambda) \times I_\Lambda$, formed of pairs (ω, σ, f) where $(\omega, \sigma) \in (\langle \Lambda \rangle / \Lambda) \times I_\Lambda$ and $f \in L_\omega^2(\mathbb{R}^d, \Lambda)$. This vector bundle is trivial, it may be identified with $(\langle \Lambda \rangle / \Lambda) \times I_\Lambda \times L^2(D_{\tilde{\Lambda}})$ just by taking the above isomorphism (note, however, that the subbundle formed of triples (ω, σ, f) such that f is smooth does not admit a smooth trivialisation).

We denote by $\mathcal{L}(\mathfrak{F})$ (resp. $\mathcal{K}(\mathfrak{F})$, $\mathcal{L}^1(\mathfrak{F})$) the vector bundles over $(\langle \Lambda \rangle / \Lambda) \times I_\Lambda$ formed of pairs (ω, σ, Q) where $Q \in \mathcal{L}(L^2(\mathbb{R}^d, \Lambda, \omega))$ (resp. $\mathcal{K}(L^2(\mathbb{R}^d, \Lambda, \omega))$, $\mathcal{L}^1(L^2(\mathbb{R}^d, \Lambda, \omega))$). Again, all these bundles are trivial. Recall that, given a Hilbert space \mathcal{H} , $\mathcal{L}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ and $\mathcal{L}^1(\mathcal{H})$ stands respectively for the space of bounded, compact and trace-class operators acting on \mathcal{H} .

Remark 3.1. *The Bloch-Floquet spectral decomposition shows that L^∞ sections of $\mathcal{L}(\mathfrak{F})$ are in one-to-one correspondence with L^∞ maps $I_\Lambda \ni \sigma \mapsto Q(\sigma) \in \mathcal{L}(L^2(\mathbb{R}^d))$, where, in addition, a.e. $Q(\sigma)$ commutes with all translations by vectors $k \in 2\pi\mathbb{Z}^d + \Lambda^\perp$.*

We denote by $\Gamma(\mathcal{K}(\mathfrak{F}))$ the space of continuous sections of $\mathcal{K}(\mathfrak{F})$. Using the previous trivialisation, it is isomorphic to $\mathcal{C}((\langle \Lambda \rangle / \Lambda) \times I_\Lambda, \mathcal{K}(L^2(D_{\tilde{\Lambda}})))$, the space of continuous functions on $(\langle \Lambda \rangle / \Lambda) \times I_\Lambda$ taking values in $\mathcal{K}(L^2(D_{\tilde{\Lambda}}))$. The dual space $\Gamma(\mathcal{K}(\mathfrak{F}))'$ to $\Gamma(\mathcal{K}(\mathfrak{F}))$ is isomorphic to $\mathcal{M}((\langle \Lambda \rangle / \Lambda) \times I_\Lambda, \mathcal{L}^1(L^2(D_{\tilde{\Lambda}})))$, the space of measures on $(\langle \Lambda \rangle / \Lambda) \times I_\Lambda$ taking values in $\mathcal{L}^1(L^2(D_{\tilde{\Lambda}}))$.

We denote by $\Gamma_+(\mathcal{K}(\mathfrak{F}))$ the subset of positive sections, and $\Gamma(\mathcal{K}(\mathfrak{F}))'_+$ the positive elements of the dual, which correspond to elements of $\mathcal{M}_+((\langle \Lambda \rangle / \Lambda) \times I_\Lambda, \mathcal{L}^1(L^2(D_{\tilde{\Lambda}})))$ (the space of measures taking values in positive trace-class operators). Note that, by the Radon-Nikodym theorem (see for instance the appendix in [21]), an element $\rho \in \Gamma_+(\mathcal{K}(\mathfrak{F}))$ can be written as $\rho = M\nu$ where $\nu = \text{Tr } \rho$ is a positive measure on $(\langle \Lambda \rangle / \Lambda) \times I_\Lambda$ and M is a ν -integrable section of $\mathcal{L}^1(\mathfrak{F})$. We shall denote by $\Gamma^1(\mathcal{L}^1(\mathfrak{F}); \nu)$ the set of such sections.

In order to state the propagation law obeyed by $\tilde{\mu}_\Lambda$ when $\mathbf{V}_h(t) = \text{Op}_h(V(t, x, \xi))$, let us introduce one more notation. Write $x = s + y \in \mathbb{R}^d$ with $(s, y) \in \Lambda^\perp \times \langle \tilde{\Lambda} \rangle$ and let $\sigma \in I_\Lambda$; we denote by $\langle V(t, y, \sigma) \rangle_\Lambda$ the average of $V(t, s + y, \sigma)$ w.r.t. s , thus getting a function that does not depend on s . We denote by $\langle V(t) \rangle_{\Lambda, \sigma}$ the multiplication operator on $L_\omega^2(\mathbb{R}^d, \Lambda)$ associated to the multiplication by the σ -depending function $\langle V(t, y, \sigma) \rangle_\Lambda$. In the trivialisation introduced above, this operator does not depend on ω . In the case of a function $a(x, \xi, \eta) \in \mathcal{S}_\Lambda^1$, the function $a(s + y, \xi, \eta)$ does not depend on s so that $\langle a(y, \sigma, \eta) \rangle_\Lambda = a(y, \sigma, \eta)$. We denote by a_σ the section of $\mathcal{L}(\mathfrak{F})$ that associates to (ω, σ) the operator acting on $L_\omega^2(\mathbb{R}^d, \Lambda)$ by $a(y, \sigma, D_y)$ (Weyl quantization). In the case of a function $a(x, \xi) \in \mathcal{C}_0^\infty(T^*\mathbb{T}^d)$ with Fourier modes in Λ (and independent of η), the function $a(s + y, \xi)$ does not depend on s so that $\langle a(y, \sigma) \rangle_\Lambda = a(y, \sigma)$. In this case a_σ is the section of $\mathcal{L}(\mathfrak{F})$ that associates to (ω, σ) the operator acting on $L_\omega^2(\mathbb{R}^d, \Lambda)$ by multiplication by $a(y, \sigma)$. Finally, $(d^2H(\sigma)D_y \cdot D_y)_\omega$ will be used to denote the operator $d^2H(\sigma)D_y \cdot D_y$ acting on $L_\omega^2(\mathbb{R}^d, \Lambda)$.

Theorem 3.2. *There exists $m_\Lambda \in \mathcal{M}_+((\langle \Lambda \rangle / \Lambda) \times I_\Lambda)$ and M_Λ^0 a m_Λ -integrable section of $\mathcal{L}^1(\mathfrak{F})$, which only depend on the sequence of initial data, such that for all $a \in \mathcal{S}_\Lambda^1$ and all $t \in \mathbb{R}$:*

$$(40) \quad \langle \tilde{\mu}_\Lambda(t, \cdot), a \rangle = \int_{(\langle \Lambda \rangle / \Lambda) \times I_\Lambda} \text{Tr}_{L_\omega^2(\mathbb{R}^d, \Lambda)}(a_\sigma M_\Lambda(t, \omega, \sigma)) m_\Lambda(d\omega, d\sigma).$$

where $M_\Lambda(t, \omega, \sigma)$ solves, m_Λ -a.e. $(\omega, \sigma) \in (\langle \Lambda \rangle / \Lambda) \times R_\Lambda$,

$$(\text{Heis}_{\Lambda, \omega, \sigma}) \quad i\partial_t M_\Lambda(t, \omega, \sigma) = \left[\frac{1}{2} (d^2 H(\sigma) D_y \cdot D_y) + \langle V(t) \rangle_{\Lambda, \sigma}, M_\Lambda(t, \omega, \sigma) \right],$$

with $M_\Lambda(0, \cdot) = M_\Lambda^0$. In other words,

$$(41) \quad M_\Lambda(t, \omega, \sigma) = U_{\Lambda, \omega, \sigma}(t) M_\Lambda^0(\omega, \sigma) U_{\Lambda, \omega, \sigma}^*(t),$$

where $U_{\Lambda, \omega, \sigma}(t)$ is the propagator starting at $t = 0$ of the unitary evolution associated to the operator $\frac{1}{2} (d^2 H(\sigma) D_y \cdot D_y) + \langle V(t) \rangle_{\Lambda, \sigma}$ on $L_\omega^2(\mathbb{R}^d, \Lambda)$.

Remark 3.3. *i) Note that the fact that $M_\Lambda(t, \cdot)$ is given by $(\text{Heis}_{\Lambda, \omega, \sigma})$ and (41) implies that it is a continuous function of t . Therefore, $\tilde{\mu}_\Lambda(t, \cdot)$ itself can be identified to a family of positive measures depending continuously on time.*

ii) The proof of Theorem 3.2 is carried out using the trivialisation obtained by identifying $L_\omega^2(\mathbb{R}^d, \Lambda)$ with $L^2(D_{\tilde{\Lambda}})$ and the final result does not depend on the choice of $\tilde{\Lambda}$ and $D_{\tilde{\Lambda}}$.

iii) Identity (40) holds when $\mathbf{V}_h(t)$ is a bounded family of perturbations as described in the introduction. In that case, the measure m_Λ may also depend on time and equation (41) is not available.

The proof of Proposition 3.2 is divided in three steps:

- (1) We first define an operator K_h which maps functions on \mathbb{R}^d to distributions with Fourier frequencies only in $\langle \Lambda \rangle$; in addition, this operator maps $(2\pi\mathbb{Z}^d)$ -periodic functions to distributions on I_Λ taking values in functions satisfying a Bloch-Floquet periodicity condition.
- (2) Then, we express $w_{I_\Lambda, h, R}(t)$ in terms of K_h and take limits, first $h \rightarrow 0^+$ then followed by $R \rightarrow +\infty$. This defines an element $\rho_\Lambda \in L^\infty(\mathbb{R}; \Gamma_+(\mathcal{K}(\mathfrak{F}))$.
- (3) We study the dependence in t of the limit object ρ_Λ and show that it obeys a Heisenberg equation similar to $(\text{Heis}_{\Lambda, \omega, \sigma})$. Note that the latter is of lower dimension than the original one (1) as soon as $\text{rk } \Lambda < d$.

Each of the next subsections is devoted to one of the steps of the proof.

3.1. First Step: Construction of the operator K_h . Take $m \in C_0^\infty((\mathbb{R}^d)^*)$ supported in the ball $B(\xi_0, \epsilon) \subset (\mathbb{R}^d)^*$, and identically equal to 1 on $B(\xi_0, \epsilon/2)$. For f a tempered distribution, we let

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} \frac{dx}{(2\pi)^{d/2}}.$$

In particular, if f is a $2\pi\mathbb{Z}^d$ -periodic function, we have $\mathcal{F}f = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \delta_k$. In what follows we shall denote $d_\Lambda := \text{rk } \Lambda$ and $d_{\Lambda^\perp} := \text{rk } \Lambda^\perp$.

The operator K_h maps a tempered distribution f to a distribution on $I_\Lambda \times \mathbb{R}^d$ as follows:

$$\begin{aligned} K_h f(\sigma, y) &:= \int_{x \in \mathbb{R}^d} f(x) \int_{\eta \in \langle \Lambda \rangle} m(\sigma) e^{\frac{i}{h} \eta \cdot y} e^{-\frac{i}{h} (\sigma + \eta) \cdot x} \frac{d\eta}{(2\pi h)^{d_\Lambda/2}} \frac{dx}{(2\pi h)^{d/2}} \\ &= \frac{m(\sigma)}{h^{d/2}} \int_{\langle \Lambda \rangle} \mathcal{F} f \left(\frac{\sigma + \eta}{h} \right) e^{\frac{i}{h} \eta \cdot y} \frac{d\eta}{(2\pi h)^{d_\Lambda/2}}. \end{aligned}$$

In order to get more insight on the properties of $K_h f$ it is useful to introduce some notations. Let $\pi_{\tilde{\Lambda}}$ be the projection on $\langle \tilde{\Lambda} \rangle$, in the direction of Λ^\perp . We have $\pi_{\tilde{\Lambda}}(2\pi\mathbb{Z}^d) = \tilde{\Lambda} \subset \mathbb{Z}^d$. For $\xi \in (\mathbb{R}^d)^*$, we shall denote by $\xi_\Lambda \in \langle \Lambda \rangle$ the linear form $\xi_\Lambda(y) := \xi \cdot \pi_{\tilde{\Lambda}}(y)$ (in other words, the projection of ξ on $\langle \Lambda \rangle$, in the direction $\tilde{\Lambda}^\perp$). Note that for $\xi \in \mathbb{Z}^d$ one has $\xi_\Lambda \in \Lambda$. We fix a bounded fundamental domain D_Λ for the action of Λ on $\langle \Lambda \rangle$. For $\eta \in \langle \Lambda \rangle$, there is a unique $\{\eta\} \in D_\Lambda$ (the “fractional part” of η) such that $\eta - \{\eta\} \in \Lambda$.

Sometimes we shall use the decomposition $(\mathbb{R}^d)^* = \tilde{\Lambda}^\perp \oplus \langle \Lambda \rangle$. This decomposition is related to the one given by the local coordinate system $F(\xi) = (\sigma(\xi), \eta(\xi))$ defined in (24) as follows. Let $(\xi_1, \xi_2) \in \tilde{\Lambda}^\perp \times \langle \Lambda \rangle$ such that F is defined on $\xi = \xi_1 + \xi_2$ (and therefore $\xi = \sigma(\xi) + \eta(\xi)$). Then $\sigma(\xi) = \sigma(\xi_1)$ does not depend on ξ_2 and

$$\xi_2 = \eta(\xi) + \sigma(\xi_1)_\Lambda.$$

In other words, $(\xi_1, \xi_2) \in \tilde{\Lambda}^\perp \times \langle \Lambda \rangle$ corresponds through F (whenever $F(\xi)$ is defined) to a pair $(\sigma, \eta) \in I_\Lambda \times \langle \Lambda \rangle$ given by:

$$(42) \quad \sigma = \sigma(\xi_1), \quad \eta = \xi_2 - \sigma(\xi_1)_\Lambda.$$

These relations imply that for every $\xi_1 \in \tilde{\Lambda}^\perp$ and $y \in \mathbb{R}^d$ the following holds:

$$(43) \quad K_h f(\sigma(\xi_1), y) = e^{-i \frac{\sigma(\xi_1)_\Lambda}{h} \cdot y} m(\sigma(\xi_1)) \int_{\Lambda^\perp} f(s + \pi_\Lambda(y)) e^{-i \frac{\xi_1}{h} \cdot s} \frac{ds}{(2\pi h)^{d_{\Lambda^\perp}/2}}.$$

In the above formula, ds is the dual density of $d\xi_1$, which in turn is defined to have $d\xi = d\xi_1 d\xi_2$ where $d\xi_2$ stands for the natural density on $\langle \Lambda \rangle$. Note that $d\xi_1$ is a constant multiple of the natural density on $\tilde{\Lambda}^\perp$.

If f is a $2\pi\mathbb{Z}^d$ -periodic function then:

$$K_h f(\sigma, y) = \frac{h^{d_{\Lambda^\perp}/2}}{(2\pi)^{d_\Lambda/2}} \sum_{k_\sigma \in \sigma(h\mathbb{Z}^d)} \delta_{k_\sigma}(\sigma) \sum_{k_\eta \in \langle \Lambda \rangle, (k_\sigma, k_\eta) \in F(h\mathbb{Z}^d)} m(k_\sigma) \hat{f}\left(\frac{k_\sigma + k_\eta}{h}\right) e^{\frac{i}{h} k_\eta \cdot y}.$$

It is clear from the above formula that for every $y \in \mathbb{R}^d$, the distribution $K_h f(\cdot, y)$ is supported on the set

$$I_\Lambda^h := \left\{ \sigma \in I_\Lambda : \frac{\sigma}{h} \in \mathbb{Z}^d - \langle \Lambda \rangle \right\}.$$

We gather the properties of the operator K_h that will be used in the sequel in the following lemma.

Lemma 3.4. (i) The Fourier transform of $K_h f(\sigma, \cdot)$ w.r.t. the second variable is:

$$(44) \quad \mathcal{F}K_h f(\sigma, \eta) = \left(\frac{2\pi}{h}\right)^{d_{\Lambda^\perp}/2} m(\sigma) \mathcal{F}u\left(\frac{\sigma}{h} + \eta\right) \delta_{\langle \Lambda \rangle}(\eta),$$

in particular it is supported in $\langle \Lambda \rangle$.

(ii) The support of $K_h f(\sigma, \cdot)$ is included in $\text{supp } f + \Lambda^\perp$.

(iii) If f is $(2\pi\mathbb{Z})^d$ -periodic, then $\text{supp } K_h f(\cdot, y) \subset I_\Lambda^h$ and $K_h f(\sigma, \cdot)$ satisfies a Floquet periodicity condition:

$$(45) \quad K_h f(\sigma, y + k) = K_h f(\sigma, y) e^{-i\omega_h(\sigma) \cdot k}$$

for all $k \in 2\pi\mathbb{Z}^d$, where

$$\omega_h : I_\Lambda^h \longrightarrow \langle \Lambda \rangle / \Lambda : \sigma \longmapsto \left\{ \frac{\sigma_\Lambda}{h} \right\}.$$

Statement (45) is equivalent to the fact that $K_h f(\sigma, \cdot)$ has only frequencies in $\Lambda - \omega_h(\sigma)$.

(iv) Let f be a $2\pi\mathbb{Z}^d$ -periodic function, and let χ be a compactly supported function on \mathbb{R}^d such that $\sum_{k \in 2\pi\mathbb{Z}^d} \chi(\cdot + k) \equiv 1$. Then

$$(46) \quad \sum_{k \in \tilde{\Lambda}} K_h(\chi f)(\sigma, y + k) e^{i\omega_h(\sigma) \cdot k} = K_h f(\sigma, y).$$

(v) If $f \in L^2(\mathbb{T}^d)$ then the following Plancherel-type formula holds:

$$(47) \quad \sum_{k \in \mathbb{Z}^d} |\hat{f}(k) m(hk)|^2 = \sum_{\sigma \in I_\Lambda^h} \int_{\mathbb{T}^d} |K_h f(\sigma, y)|^2 dy.$$

Proof. All points are quite obvious except (iii), which we prove below. Formula (44) shows that the Fourier transform of $K_h f(\sigma, \cdot)$ is supported in $\langle \Lambda \rangle$. On the other hand, if f is $2\pi\mathbb{Z}^d$ -periodic then its Fourier transform is supported in \mathbb{Z}^d . Therefore, because of identity (44), on the support of $\mathcal{F}K_h f(\sigma, \eta)$ one must have:

$$\frac{\sigma}{h} + \eta \in \mathbb{Z}^d, \quad \eta \in \langle \Lambda \rangle.$$

In other words, $\sigma \in I_\Lambda^h$ and $\eta \in \langle \Lambda \rangle$. Taking the projection on $\langle \Lambda \rangle$ in the direction $\tilde{\Lambda}^\perp$ yields $\eta \in \Lambda - \frac{\sigma_\Lambda}{h}$, which is equivalent to $\eta \in \Lambda - \left\{ \frac{\sigma_\Lambda}{h} \right\}$. Note that any other choice of the auxiliary lattice used to define the projection onto $\langle \Lambda \rangle$ would lead to a $\sigma'_\Lambda \in \langle \Lambda \rangle$ that differs from σ_Λ on an element of $h\Lambda$. This shows, in particular, that the mapping ω_h is well-defined on I_Λ^h . \square

Remark 3.5. Let f be $2\pi\mathbb{Z}^d$ -periodic and let $\theta \in (\mathbb{R}^d)^*/(\mathbb{Z}^d)^*$. Let $g_\theta(y) := e^{-i\theta \cdot y} f(y)$; then the proof of Lemma 3.4 (iii) shows that $K_h g_\theta$ satisfies the following Bloch-Floquet periodicity condition:

$$K_h g_\theta(\sigma, y + k) = K_h g_\theta(\sigma, y) e^{-i(\omega_h(\sigma) + \theta_\Lambda) \cdot k},$$

for every $k \in 2\pi\mathbb{Z}^d$.

3.2. Second step: Link between $w_{I_\Lambda, h, R}$ and K_h . Now we show how the two-microlocal Wigner distributions $(w_{I_\Lambda, h, R})$ of the sequence $(S_h^{t/h} u_h)$ can be expressed in terms of $(K_h S_h^{t/h} u_h)$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ be such that $\sum_{k \in 2\pi\mathbb{Z}^d} \chi(\cdot + k) \equiv 1$. All the following identities hold independently of the choice of such χ . We start expressing the standard Wigner distributions $w_{u_h}^h$ in terms of the decomposition $\xi = (\xi_1, \xi_2) \in \tilde{\Lambda}^\perp \times \langle \Lambda \rangle$ of $(\mathbb{R}^d)^*$. Let $b \in \mathcal{C}_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, possibly depending on h . The following holds:

$$\begin{aligned} \mathcal{I}(b, h) &:= \int_{T^*\mathbb{T}^d} b(x, \xi) w_{u_h}^h(dx, d\xi) \\ &= \frac{1}{(2\pi)^{d/2}} \int_{(\mathbb{R}^d)^* \times (\mathbb{R}^d)^*} \mathcal{F}(\chi u_h)(\xi) \overline{\mathcal{F}u_h(\xi')} \mathcal{F}b\left(\xi' - \xi, h \frac{\xi + \xi'}{2}\right) d\xi d\xi' \\ &= \frac{1}{(2\pi)^{d/2} h^{d_{\Lambda^\perp}}} \int_{\xi_1, \xi'_1 \in \tilde{\Lambda}^\perp, \xi_2, \xi'_2 \in \langle \Lambda \rangle} \mathcal{F}(\chi u_h)(\xi_1, \xi_2) \overline{\mathcal{F}u_h(\xi'_1, \xi'_2)} \\ &\quad \mathcal{F}b\left(\frac{\xi'_1 - \xi_1}{h} + \xi'_2 - \xi_2, \frac{\xi_1 + \xi'_1}{2} + h \frac{\xi_2 + \xi'_2}{2}\right) d\xi_1 d\xi'_1 d\xi_2 d\xi'_2. \end{aligned}$$

Identity (42) shows that:

$$\xi = \frac{\xi_1}{h} + \xi_2 = \frac{\sigma}{h} + \eta,$$

where $\sigma = \sigma(\xi_1)$ and $\eta = \xi_2 - \frac{\sigma(\xi_1)}{h}$. For every $\sigma \in I_\Lambda$, we denote by $\xi_1(\sigma)$ the element in $\tilde{\Lambda}^\perp$ characterised by $\xi_1(\sigma) - \sigma \in \langle \Lambda \rangle$. The density $d\xi_1$ on $\tilde{\Lambda}^\perp$ is transferred to a density $d\sigma$ on I_Λ (note that $d\xi = d\xi_1 d\xi_2 = d\sigma d\eta$).

With this in mind, we obtain using (44), provided we assume that u_h has frequencies in $B(\xi_0, \epsilon/2)$:

$$\begin{aligned} (48) \quad \mathcal{I}(b, h) &= \frac{(2\pi h)^{-d_{\Lambda^\perp}}}{(2\pi)^{d/2}} \int_{\sigma, \sigma' \in I_\Lambda, \eta, \eta' \in \mathbb{R}^d} \mathcal{F}K_h \chi u_h(\sigma, \eta) \overline{\mathcal{F}K_h u_h(\sigma', \eta')} \\ &\quad \mathcal{F}b\left(\frac{\xi_1(\sigma') - \xi_1(\sigma)}{h} + \frac{\sigma'_\Lambda}{h} + \eta' - \frac{\sigma_\Lambda}{h} - \eta, \frac{\sigma \oplus \sigma'}{2} - \left(\frac{\sigma \oplus \sigma'}{2}\right)_\Lambda + \frac{\sigma_\Lambda + \sigma'_\Lambda}{2} + h \frac{\eta + \eta'}{2}\right) \\ &\quad d\sigma d\sigma' d\eta d\eta', \end{aligned}$$

where $\frac{\sigma \oplus \sigma'}{2}$ is a notation for the image under the coordinate map $\sigma(\xi)$ of $\frac{\sigma + \sigma'}{2}$.

If b is invariant in the direction Λ^\perp then the previous integral reduces to an integral over $\sigma = \sigma'$ and takes the simpler form

$$\mathcal{I}(b, h) = \frac{(2\pi h)^{-d_{\Lambda^\perp}}}{(2\pi)^{d_{\Lambda/2}}} \int_{\sigma \in I_\Lambda, \eta, \eta' \in \mathbb{R}^d} \mathcal{F}K_h \chi u_h(\sigma, \eta) \overline{\mathcal{F}K_h u_h(\sigma, \eta')} \mathcal{F}b\left(\eta' - \eta, \sigma + h \frac{\eta + \eta'}{2}\right) d\sigma d\eta d\eta',$$

where now $\mathcal{F}b$ should just be interpreted as a partial Fourier transform in the direction $\langle \tilde{\Lambda} \rangle$.

In (48), the function $K_h \chi u_h$ may be replaced by any other function satisfying the identity (46). In particular, we may replace $K_h \chi u_h$ by $\chi_0 K_h u_h$, where χ_0 is a function that satisfies

$$\sum_{k \in \tilde{\Lambda}} \chi_0(\cdot + k) \equiv 1$$

and χ_0 is constant in the direction Λ^\perp . In what follows, we take χ_0 to be the characteristic functions of $D_{\tilde{\Lambda}}$, a fundamental domain for the action of $\tilde{\Lambda}$ on \mathbb{R}^d .

By the changes of variables $\eta \longrightarrow \eta - \frac{\sigma_\Lambda}{h} + \frac{1}{h} \left(\frac{\sigma \oplus \sigma'}{2} \right)_\Lambda$ and $\eta' \longrightarrow \eta' - \frac{\sigma'_\Lambda}{h} + \frac{1}{h} \left(\frac{\sigma \oplus \sigma'}{2} \right)_\Lambda$, (48) may be transformed into

$$(49) \quad \mathcal{I}(b, h) = \frac{(2\pi h)^{-d_{\Lambda^\perp}}}{(2\pi)^{d/2}} \int_{\sigma, \sigma' \in I_\Lambda, \eta, \eta' \in \mathbb{R}^d} \mathcal{F} \chi_0 K_h u_h \left(\sigma, \eta - \frac{\sigma_\Lambda}{h} + \frac{1}{h} \left(\frac{\sigma \oplus \sigma'}{2} \right)_\Lambda \right) \\ \times \overline{\mathcal{F} K_h u_h \left(\sigma', \eta' - \frac{\sigma'_\Lambda}{h} + \frac{1}{h} \left(\frac{\sigma \oplus \sigma'}{2} \right)_\Lambda \right)} \\ \times \mathcal{F} b \left(\frac{\xi_1(\sigma') - \xi_1(\sigma)}{h} + \eta' - \eta, \frac{\sigma \oplus \sigma'}{2} + h \frac{\eta + \eta'}{2} \right) d\sigma d\sigma' d\eta d\eta'.$$

Next we write

$$K_h u_h(\sigma, y) = \sum_{k \in \tilde{\Lambda}} \chi_0(y + k) K_h u_h(\sigma, y) = \sum_{k \in \tilde{\Lambda}} (\chi_0 K_h u_h)(\sigma, y + k) e^{i\omega_h(\sigma) \cdot k},$$

so that

$$\mathcal{F} K_h u_h(\sigma, \eta) = \mathcal{F} \chi_0 K_h u_h(\sigma, \eta) \delta_{\Lambda - \omega_h(\sigma) + \tilde{\Lambda}^\perp}(\eta),$$

Thus (49) is also

$$(50) \quad \mathcal{I}(b, h) = \frac{(2\pi h)^{-d_{\Lambda^\perp}}}{(2\pi)^{d/2}} \int_{\sigma, \sigma' \in I_\Lambda, \eta, \eta' \in \mathbb{R}^d} \mathcal{F} \chi_0 K_h u_h \left(\sigma, \eta - \frac{\sigma_\Lambda}{h} + \frac{1}{h} \left(\frac{\sigma \oplus \sigma'}{2} \right)_\Lambda \right) \\ \times \overline{\mathcal{F} \chi_0 K_h u_h \left(\sigma', \eta' - \frac{\sigma'_\Lambda}{h} + \frac{1}{h} \left(\frac{\sigma \oplus \sigma'}{2} \right)_\Lambda \right)} \\ \times \mathcal{F} b \left(\frac{\xi_1(\sigma') - \xi_1(\sigma)}{h} + \eta' - \eta, \frac{\sigma \oplus \sigma'}{2} + h \frac{\eta + \eta'}{2} \right) d\sigma d\sigma' d\eta d\eta' \delta_{\eta' \in \tilde{\Lambda}^\perp + \omega_h(\frac{\sigma \oplus \sigma'}{2})}.$$

This motivates the following definition :

Definition 3.6. If $Q(\omega, s, \sigma)$ is a smooth compactly supported function on $\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda$, taking values in $\mathcal{L}(L^2(D_{\tilde{\Lambda}}))$, we define:

$$P_Q^h(s, \sigma) := e^{i\frac{\sigma_\Lambda}{h}} Q(\omega_h(\sigma), s, \sigma) e^{-i\frac{\sigma_\Lambda}{h}}, \quad v_h(\sigma, y) := \chi_0(y) e^{i\frac{\sigma_\Lambda}{h} \cdot y} K_h u_h(\sigma, y),$$

where $e^{i\frac{\sigma_\Lambda}{h}}$ denotes multiplication by $e^{i\frac{\sigma_\Lambda}{h} \cdot y}$. Define

$$(51) \quad \langle \rho_{u_h}^h, Q \rangle := \langle v_h, P_Q^h(hD_\sigma, \cdot) v_h \rangle_{L^2(I_\Lambda; L^2(D_{\tilde{\Lambda}}))},$$

where $P_Q^h(hD_\sigma, \sigma)$ is obtained from P_Q^h by Weyl quantization.

More explicitly, we have:

$$(52) \quad \langle \rho_{u_h}^h, Q \rangle = \frac{1}{(2\pi h)^{d_{\Lambda^\perp}}} \int_{\sigma, \sigma' \in I_\Lambda} \int_{s \in \Lambda^\perp} e^{-i \frac{\xi_1(\sigma') - \xi_1(\sigma)}{h} \cdot s} \left\langle v_h(\sigma', \cdot), P_Q^h \left(s, \frac{\sigma \oplus \sigma'}{2} \right) v_h(\sigma', \cdot) \right\rangle_{L^2(D_{\tilde{\Lambda}})} d\sigma d\sigma' ds.$$

The interest of this definition becomes clearer if we realize the following identity. Let $b \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$, then formula (50) is equivalent to the identity

$$(53) \quad \langle u_h, \text{Op}_h(b) u_h \rangle_{L^2(\mathbb{T}^d)} = \langle \rho_{u_h}^h, Q_b^h \rangle,$$

where $Q_b^h(\omega, s, \sigma)$ is the operator on $L^2(D_{\tilde{\Lambda}})$ given by the kernel

$$(54) \quad Q_b^h(\omega, s, \sigma)(\tilde{y}', \tilde{y}) = \frac{1}{(2\pi)^{d_\Lambda}} \sum_{k \in \tilde{\Lambda}} e^{i\omega k} \int_{\eta \in \langle \Lambda \rangle} b \left(s + \frac{\tilde{y} + \tilde{y}'}{2}, \sigma + h\eta \right) e^{i\eta \cdot (\tilde{y}' - \tilde{y} + k)} d\eta,$$

where, recall we have written $x = s + \tilde{y} \in \Lambda^\perp \oplus \langle \tilde{\Lambda} \rangle$. Note that if one identifies $L^2(D_{\tilde{\Lambda}})$ with $L_\omega^2(\mathbb{R}^d, \Lambda)$ as we did before, the operator $Q_b^h(\omega, s, \sigma)$ then corresponds to the (ω -independent) Weyl pseudodifferential operator $b(y, \sigma + hD_y)$ acting on $L_\omega^2(\mathbb{R}^d, \Lambda)$.

Let us now consider $a \in \mathcal{S}_\Lambda^1$ and let $b_{h,R}(x, \xi) := a(x, \xi, \eta(\xi)/h) \chi(\eta(\xi)/hR)$. We then have

$$(55) \quad \begin{aligned} \langle S_h^{t/h} u_h, \text{Op}_h(b_h) S_h^{t/h} u_h \rangle_{L^2(\mathbb{T}^d)} &= \langle w_{I_\Lambda, h, R}(t), a \rangle \\ &= \langle \rho^h(t), Q_{a,R}^h \rangle + O_R(h) \end{aligned}$$

where $\rho^h(t) := \rho_{S_h^{t/h} u_h}^h$ and $Q_{a,R}^h := Q_{b_{h,R}}^h$. Note that $Q_{a,R}^h$ does not depend on s , since as a has only frequencies in Λ it is a function independent on s .

We now take limits as h tends to zero :

Proposition 3.7. *After extraction of a subsequence, there exist*

$$\rho_\Lambda \in L^\infty(\mathbb{R}_t, \mathcal{D}'(\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda; \mathcal{L}^1(L^2(D_{\tilde{\Lambda}}))))$$

such that for every $Q \in \mathcal{C}_c^\infty(\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda; \mathcal{K}(L^2(D_{\tilde{\Lambda}})))$ and every $\phi \in L^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \phi(t) \langle \rho^h(t), Q \rangle dt \longrightarrow \int_{\mathbb{R}} \phi(t) \langle \rho_\Lambda(t), Q \rangle dt.$$

In addition, ρ_Λ is positive when restricted to symbols $Q(\omega, s, \sigma)$ that do not depend on s .

Proof. Note that Lemma 3.4, v) implies that (v_h) is bounded in $L^2(I_\Lambda; L^2(D_{\tilde{\Lambda}}))$ and that the Calderón-Vaillancout theorem gives that the operators $P_Q^h(hD_\sigma, \sigma)$ are uniformly bounded with respect to h . Therefore, the linear map

$$L_h : Q \mapsto \int_{\mathbb{R}} \langle \rho^h(t), Q(t) \rangle dt$$

is uniformly bounded as $h \longrightarrow 0$. Therefore, for any Q , up to extraction of a subsequence, it has a limit $l(Q)$.

Considering a countable dense subset of $L^1(\mathbb{R}; \mathcal{C}_c^\infty(\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda; \mathcal{K}(L^2(D_{\tilde{\Lambda}})))$, and using a diagonal extraction process, one finds a sequence (h_n) tending to 0 as n goes to $+\infty$ such that for any $Q \in L^1(\mathbb{R}; \mathcal{C}_c^\infty(\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda; \mathcal{K}(L^2(D_{\tilde{\Lambda}})))$, the sequence $L_{h_n}(Q)$ has a limit as n goes to $+\infty$.

The limit is a linear form on $L^1(\mathbb{R}; \mathcal{C}_c^\infty(\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda; \mathcal{K}(L^2(D_{\tilde{\Lambda}})))$, characterized by an element ρ_Λ of the dual bundle $L^\infty(\mathbb{R}_t, \mathcal{D}'(\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda; \mathcal{L}^1(L^2(D_{\tilde{\Lambda}})))$. Finally, note that if $Q(t, \omega, \sigma)$ is a positive operator independent of s , equation (51) gives $L_h(Q) \geq 0$, whence the positivity of ρ_Λ when restricted to symbols that do not depend on s . \square

As a consequence and in view of (55), letting h going to 0, then R to $+\infty$, we have (possibly along a subsequence) for every $a \in \mathcal{S}_\Lambda^1$ and $\phi \in L^1(\mathbb{R})$:

$$\begin{aligned} \lim_{R \rightarrow +\infty} \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \phi(t) \langle w_{I_\Lambda, h, R}(t), a \rangle dt &= \lim_{R \rightarrow +\infty} \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \phi(t) \langle \rho^h(t), Q_{a, R} \rangle dt \\ &= \lim_{R \rightarrow +\infty} \int_{\mathbb{R}} \phi(t) \langle \rho_\Lambda(t), Q_{a, R} \rangle dt \\ &= \int_{\mathbb{R}} \phi(t) \langle \rho_\Lambda(t), Q_{a, \infty} \rangle dt, \end{aligned}$$

where $Q_{a, \infty}(\omega, s, \sigma)$ is the bounded operator on $L^2(D_{\tilde{\Lambda}})$ given by the kernel

$$(56) \quad Q_{a, \infty}(\omega, s, \sigma)(\tilde{y}', \tilde{y}) = \frac{1}{(2\pi)^{d_\Lambda}} \sum_{k \in \tilde{\Lambda}} e^{i\omega k} \int_{\eta \in \langle \Lambda \rangle} a\left(s + \frac{\tilde{y} + \tilde{y}'}{2}, \sigma, \eta\right) e^{i\eta \cdot (\tilde{y}' - \tilde{y} + k)} d\eta.$$

As discussed before, the operator $Q_{a, \infty}$ corresponds to the Weyl operator $a(s + y, \sigma, D_y)$ acting on $L_\omega^2(\mathbb{R}^d \langle \Lambda \rangle)$. In particular, when $a \in \mathcal{C}_c^\infty(\mathbb{T}^d)$ has only frequencies in Λ , the operator $Q_{a, \infty}$ is the multiplication operator a_σ appearing in identity (40) of Theorem 3.2.

At this stage of the analysis, we have completed the proof of the first part of Theorem 3.2 (equation (40)), using only the fact that $\mathbf{V}_h(t)$ is a bounded perturbation : we let $m_\Lambda(t, \omega, s, \sigma) := \text{Tr}_{L_\omega^2(\mathbb{R}^d, \Lambda)} \rho_\Lambda(t, \omega, s, \sigma)$. We have $\rho_\Lambda = M_\Lambda m_\Lambda$ where $\text{Tr} M_\Lambda = 1$.

As already noted, equation (40) implies the absolute continuity result, Theorem 1.3 (2).

3.3. Step 3: Showing a propagation law. From now on we shall assume $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ and prove the propagation law $(\text{Heis}_{\Lambda, \omega, \sigma})$. The first crucial observation is the following lemma.

Lemma 3.8. *The measure ρ_Λ is invariant by the Hamiltonian flow. More precisely, for every $Q \in \mathcal{C}_c^\infty(\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda, \mathcal{K}(L^2(D_{\tilde{\Lambda}})))$ and a.e. t ,*

$$\langle \rho_\Lambda(t), dH(\sigma) \cdot \partial_s Q \rangle = 0.$$

In particular, the restriction of $\rho_\Lambda(t)$ to $\sigma \in R_\Lambda$ is invariant under the action of Λ^\perp by translation on the parameter s .

Proof. This lemma may be understood as a consequence of the invariance by the classical flow of semi-classical measures. Indeed, the same arguments than those of Appendix 8 give that for all $\ell \in \mathbb{R}$, we have

$$\langle w_{I_\Lambda, h, R}(t), a \rangle = \langle w_{I_\Lambda, h, R}(t), a \circ \phi_\ell^0 \rangle + o(1)$$

as h goes to 0 (R fixed). As a consequence, we deduce that for all $\ell \in \mathbb{R}$,

$$\langle \rho^h(t), Q_{a, R} \rangle = \langle \rho^h(t), Q_{a \circ \phi_\ell^0, R} \rangle + o(1)$$

as h goes to 0 and R to $+\infty$. Recall that $\phi_\ell^0(x, \xi) = (x + \ell dH(\xi), \xi)$ and that for $\sigma \in I_\Lambda$, $dH(\sigma) \in \Lambda^\perp$. As a consequence, if $x = s + y \in \Lambda^\perp \oplus \langle \tilde{\Lambda} \rangle$, then for $\sigma \in I_\Lambda$, the vector $x + \ell dH(\sigma)$ decomposes as

$$x + \ell dH(\sigma) = (s + \ell dH(\sigma)) + y \in \Lambda^\perp \oplus \langle \tilde{\Lambda} \rangle,$$

and the kernel of the operator $Q_{a \circ \phi_\ell^0, R}(\omega, s, \sigma)$ is the function

$$\begin{aligned} Q_{a \circ \phi_\ell^0, R}(\omega, s, \sigma)(\tilde{y}', \tilde{y}) &= \frac{1}{(2\pi)^{d_\Lambda}} \sum_{k \in \tilde{\Lambda}} e^{i\omega k} \int a \left(s + \ell dH(\sigma) + \frac{\tilde{y} + \tilde{y}'}{2}, \sigma, \eta \right) \\ &\quad \times \chi(\eta/R) e^{i\eta \cdot (\tilde{y}' - \tilde{y} + k)} d\eta. \end{aligned}$$

The result follows if we note that compact pseudodifferential operators (e.g. operators of the form $Q_{a, R}(\omega, s, \sigma)$) are dense in $\mathcal{K}(L^2(D_{\tilde{\Lambda}}))$ for the weak topology of operators. \square

We finally show that ρ_Λ , restricted to $\sigma \in R_\Lambda$ obeys the propagation law ($\text{Heis}_{\Lambda, \omega, \sigma}$). From now on, we only consider test symbols $Q(\omega, \sigma)$ that do not depend on the parameter $s \in \Lambda^\perp$. We recall that we write $V(t, x, \xi)$ as $V(t, s + \tilde{y}, \sigma + \eta)$ and that we use the notation $\langle V(t, \tilde{y}, \sigma + \eta) \rangle_\Lambda$ to mean that we are averaging $V(t, s + \tilde{y}, \sigma + \eta)$ w.r.t. s , thus getting a function that does not depend on s (nor η when $\xi \in I_\Lambda$). In the notation of (54), we note that $Q_{\langle V(t, \cdot, \sigma) \rangle_\Lambda}^{h=0} = \langle V(t) \rangle_{\Lambda, \sigma}$ defines a multiplication operator on $L^2(D_{\tilde{\Lambda}})$ (for which the Floquet-Bloch periodicity conditions are transparent). To simplify the notation, we set in what follows:

$$A(\sigma, \eta) := \frac{1}{2} d^2 H(\sigma) \eta \cdot \eta.$$

To prove that ρ_Λ satisfies a Schrödinger-type equation, we note that

$$\begin{aligned} K_h H(hD_x) f(\sigma, y) &= \frac{1}{h^{d/2}} m(\sigma) \int_{\eta \in \langle \Lambda \rangle} H(\sigma + \eta) \mathcal{F} f \left(\frac{\sigma + \eta}{h} \right) e^{\frac{i}{h} \eta \cdot y} \frac{d\eta}{(2\pi h)^{d_\Lambda/2}} \\ &= H(\sigma + hD_y) K_h f(\sigma, y), \end{aligned}$$

and that

$$K_h \text{Op}_h(V(t, \cdot)) f(\sigma, y) = P_{Q_V^h}^h(hD_\sigma, \sigma) K_h f(\sigma, y)$$

(we used the notation of Definition 3.6). Therefore, $w_h(t, \cdot) := K_h S_h^{t/h} u_h$ solves:

$$i\partial_t w_h(t, \sigma, y) = \left(h^{-2} H(\sigma + hD_y) + P_{Q_V^h}^h(hD_\sigma, \sigma) \right) w_h(t, \sigma, y).$$

Note that if $Q(\omega, s, \sigma)$ does not depend on s , we have

$$\begin{aligned} & \langle \rho^h(t), [h^{-2}H(\sigma + hD_y), Q] \rangle \\ &= \langle \chi_0 e^{i\frac{\sigma\Lambda}{h}} (h^{-2}H(\sigma + hD_y)) w_h(t), P_Q^h(hD_\sigma, \cdot) \chi_0 e^{i\frac{\sigma\Lambda}{h}} w_h(t) \rangle \\ & \quad - \langle \chi_0 e^{i\frac{\sigma\Lambda}{h}} w_h(t), P_Q^h(hD_\sigma, \cdot) \chi_0 e^{i\frac{\sigma\Lambda}{h}} (h^{-2}H(\sigma + hD_y)) w_h(t) \rangle_{L^2(I_\Lambda; L^2(D_\Lambda))} \end{aligned}$$

Hence, passing to the limit $h \rightarrow 0$,

$$\int_{\mathbb{R}} \phi'(t) \langle \rho^h(t), Q \rangle dt = i \int_{\mathbb{R}} \phi(t) \langle \rho^h(t), [h^{-2}H(\sigma + hD_y)_\omega + Q_V^h(s, \sigma), Q] \rangle dt + o(1).$$

Above, the index ω in $H(\sigma + hD_y)_\omega$ indicates that the operator acts on $L^2(\langle D_\Lambda \rangle)$ with Floquet periodicity conditions (45). We perform a Taylor expansion of $H(\sigma + hD_y)$ and write, in $\mathcal{L}(L^2(\langle D_\Lambda \rangle))$, for any $Q \in \mathcal{K}(L^2(\langle D_\Lambda \rangle))$,

$$H(\sigma + hD_y)Q = H(\sigma)Q + h dH(\sigma)D_y Q + h^2 A(\sigma, D_y)Q + O(h^3).$$

At this point, note that $dH(\sigma)D_y = 0$ (since $\sigma \in I_\Lambda$ one has $dH(\sigma) \in \Lambda^\perp$). Therefore, for $Q \in \mathcal{C}_c^\infty(\langle \Lambda \rangle / \Lambda \times \Lambda^\perp \times I_\Lambda, \mathcal{K}(L^2(D_\Lambda)))$,

$$[h^{-2}H(\sigma + hD_y)_\omega, Q(\omega, hD_\sigma, \sigma)] = [A(\sigma, D_y)_\omega, Q(\omega, hD_\sigma, \sigma)] + O(h).$$

As a consequence, we obtain:

$$\int_{\mathbb{R}} \phi'(t) \langle \rho^h(t), Q \rangle dt = i \int_{\mathbb{R}} \phi(t) \langle \rho^h(t), [A(\sigma, D_y)_\omega + Q_V^h(hD_\sigma, \sigma), Q(\omega, hD_\sigma, \sigma)] \rangle dt + o(1).$$

Taking limits, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \phi'(t) \langle \rho_\Lambda(t), Q \rangle dt &= i \int_{\mathbb{R}} \phi(t) \langle \rho_\Lambda(t), [A(\sigma, D_y)_\omega + Q_V^0(s, \sigma), Q] \rangle dt \\ &= i \int_{\mathbb{R}} \langle \rho_\Lambda(t), [A(\sigma, D_y)_\omega + Q_{\langle V \rangle}^0(s, \sigma), Q] \rangle dt \end{aligned}$$

where the potential $\langle V \rangle(s, \sigma)$ is averaged along the flow $s \mapsto s + tdH(\sigma)$ (because of Lemma 3.8). But $\langle V \rangle$ does not depend on s for $\sigma \in R_\Lambda$, and it is simply the average of V w.r.t. s . Hence $Q_{\langle V \rangle}^0 = \langle V(t) \rangle_{\Lambda, \sigma}$ and ρ_Λ satisfies the following Heisenberg equation for $\sigma \in R_\Lambda$:

$$i\partial_t \rho_\Lambda(t, \omega, \sigma) = [A(\sigma, D_y)_\omega + \langle V(t) \rangle_{\Lambda, \sigma}, \rho_\Lambda(t, \omega, \sigma)]$$

(note that $\rho_\Lambda(t, \omega, \sigma)$ does not depend on s for $\sigma \in R_\Lambda$). Let

$$m_\Lambda(t, \omega, s, \sigma) := \text{Tr}_{L_\omega^2(\mathbb{R}^d, \Lambda)} \rho_\Lambda(t, \omega, s, \sigma);$$

the propagation law (57) implies that m_Λ does not depend on t . Therefore, $\rho_\Lambda = M_\Lambda m_\Lambda$ where $M_\Lambda(\cdot, \omega, \sigma)$ solves (Heis $_{\Lambda, \omega, \sigma}$) for $\sigma \in R_\Lambda$ and $\text{Tr} M_\Lambda = 1$. This concludes the proof of Theorem 3.2 (in the statement, the parameter s disappeared since all test functions are independent of s).

4. AN ITERATIVE PROCEDURE FOR COMPUTING μ

In this section, we develop the iterative procedure which leads to the proof of Theorem 1.10

4.1. First step of the construction. What was done in the previous section can be considered as the first step of an iterative procedure that allows to effectively compute the measure $\mu(t, \cdot)$ solely in terms of the sequence of initial data (u_h) . Recall that we assumed in §2.2, without loss of generality, that the projection on ξ of $\mu(t, \cdot)$ was supported in a ball contained in $\mathbb{R}^d \setminus C_H$. We have decomposed this measure as a sum

$$\mu(t, \cdot) = \sum_{\Lambda \in \mathcal{L}} \mu_\Lambda(t, \cdot) + \sum_{\Lambda \in \mathcal{L}} \mu^\Lambda(t, \cdot),$$

where Λ runs over the set of primitive submodules of \mathbb{Z}^d , and where

$$\mu_\Lambda(t, \cdot) = \int_{\langle \Lambda \rangle} \tilde{\mu}_\Lambda(t, \cdot, d\eta) \rfloor_{\mathbb{T}^d \times R_\Lambda}, \quad \mu^\Lambda(t, \cdot) = \int_{\langle \Lambda \rangle} \tilde{\mu}^\Lambda(t, \cdot, d\eta) \rfloor_{\mathbb{T}^d \times R_\Lambda}.$$

From Theorem 2.6, the distributions $\tilde{\mu}_\Lambda$ have the following properties :

- (1) $\tilde{\mu}_\Lambda(t, dx, d\xi, d\eta)$ is in $\mathcal{C}(\mathbb{R}; (\mathcal{S}_\Lambda^1)')$ and all its x -Fourier modes are in Λ ; with respect to the variable ξ , $\tilde{\mu}_\Lambda(t, dx, d\xi, d\eta)$ is supported in I_Λ ;
- (2) if $\tau_h \ll 1/h$ then for every $t \in \mathbb{R}$, $\tilde{\mu}_\Lambda(t, \cdot)$ is a positive measure and:

$$\tilde{\mu}_\Lambda(t, \cdot) = \left(\tilde{\phi}_t^1 \right)_* \tilde{\mu}_\Lambda(0, \cdot),$$

where:

$$\tilde{\phi}_s^1 : (x, \xi, \eta) \longmapsto (x + sd^2 H(\sigma(\xi))\eta, \xi, \eta);$$

- (3) if $\tau_h = 1/h$ then $\int_{\langle \Lambda \rangle} \tilde{\mu}_\Lambda(t, \cdot, d\eta)$ is in $\mathcal{C}(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$ and $\int_{\mathbb{R}^d \times \langle \Lambda \rangle} \tilde{\mu}_\Lambda(t, \cdot, d\xi, d\eta)$ is an absolutely continuous measure on \mathbb{T}^d . In fact, with the notations of Section 2.4, we have, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ with Fourier modes in Λ ,

$$\int_{\mathbb{T}^d \times I_\Lambda \times \langle \Lambda \rangle} a(x, \xi) \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta) = \int_{(\langle \Lambda \rangle / \Lambda) \times I_\Lambda} \text{Tr}(a_\sigma \rho_\Lambda(t, d\omega, d\sigma))$$

where $\rho_\Lambda \in L^\infty(\mathbb{R}_t, \Gamma(\mathcal{K}(\mathfrak{F}))'_+)$ and a_σ is the section of $\mathcal{L}(\mathfrak{F})$ defined by the map $(\omega, \sigma) \mapsto$ multiplication by $a(y, \sigma)$. In addition, if $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ then $\rho_\Lambda = M_\Lambda m_\Lambda$ where $m_\Lambda \in \mathcal{M}_+((\langle \Lambda \rangle / \Lambda) \times I_\Lambda)$, M_Λ is a section of $\mathcal{L}^1(\mathfrak{F})$ integrable with respect to m_Λ . Moreover, $\text{Tr}_{L_\omega^2(\mathbb{R}^d, \Lambda)} M_\Lambda(t, \omega, \sigma) = 1$ and $M_\Lambda(\cdot, \omega, \sigma)$ satisfies a Heisenberg equation $(\text{Heis}_{\Lambda, \omega, \sigma})$.

On the other hand, the measures $\tilde{\mu}^\Lambda$ satisfy:

- (1) for $a \in \mathcal{S}_\Lambda^1$, $\langle \tilde{\mu}^\Lambda(t, dx, d\xi, d\eta), a(x, \xi, \eta) \rangle$ is obtained as the limit of

$$\langle w_{h, R, \delta}^{I_\Lambda}(t), a \rangle = \int_{T^*\mathbb{T}^d} \chi\left(\frac{\eta(\xi)}{\delta}\right) \left(1 - \chi\left(\frac{\tau_h \eta(\xi)}{R}\right)\right) a(x, \xi, \tau_h \eta(\xi)) w_h(t)(dx, d\xi),$$

in the weak-* topology of $L^\infty(\mathbb{R}, (\mathcal{S}_\Lambda^1)')$, as $h \rightarrow 0^+$, $R \rightarrow +\infty$ and then $\delta \rightarrow 0^+$ (possibly along subsequences);

- (2) $\tilde{\mu}^\Lambda(t, dx, d\xi, d\eta)$ is in $L^\infty(\mathbb{R}, \mathcal{M}_+(T^*\mathbb{T}^d \times \overline{\langle \Lambda \rangle}))$ and all its x -Fourier modes are in Λ . With respect to the variable η , the measure $\tilde{\mu}^\Lambda(t, dx, d\xi, d\eta)$ is 0-homogeneous and supported at infinity : we see it as a measure on the sphere at infinity $\mathbb{S}\langle \Lambda \rangle$. With respect to the variable ξ , it is supported on $\{\xi \in I_\Lambda\}$;

- (3) $\tilde{\mu}^\Lambda$ is invariant by the two flows,

$$\phi_s^0 : (x, \xi, \eta) \mapsto (x + sdH(\xi), \xi, \eta), \quad \text{and} \quad \phi_s^1 : (x, \xi, \eta) \mapsto (x + sd^2H(\sigma(\xi)) \frac{\eta}{|\eta|}, \xi, \eta).$$

This is the first step of an iterative procedure; the next step is to decompose the measure $\mu^\Lambda(t, \cdot)$ according to primitive submodules of Λ . We need to adapt the discussion of [3]; to this aim, we introduce some additional notation.

Fix a primitive submodule $\Lambda \subseteq \mathbb{Z}^d$ and $\sigma \in I_\Lambda \setminus C_H$. For $\Lambda_2 \subseteq \Lambda_1 \subseteq \Lambda$ primitive submodules of $(\mathbb{Z}^d)^*$, for $\eta \in \langle \Lambda_1 \rangle$, we denote

$$\begin{aligned} \Lambda_\eta(\sigma, \Lambda_1) &:= (\Lambda_1^\perp \oplus \mathbb{R} d^2H(\sigma).\eta)^\perp \cap (\mathbb{Z}^d)^* \\ &= (\mathbb{R} d^2H(\sigma).\eta)^\perp \cap \Lambda_1, \end{aligned}$$

where the orthogonal is always taken in the sense of duality. We note that $\Lambda_\eta(\sigma, \Lambda_1)$ is a primitive submodule of Λ_1 , and that the inclusion $\Lambda_\eta(\sigma, \Lambda_1) \subset \Lambda_1$ is strict if $\eta \neq 0$ since $d^2H(\sigma)$ is definite. We define:

$$R_{\Lambda_2}^{\Lambda_1}(\sigma) := \{\eta \in \langle \Lambda_1 \rangle, \Lambda_\eta(\sigma, \Lambda_1) = \Lambda_2\}.$$

Because $d^2H(\sigma)$ is definite, we have the decomposition $(\mathbb{R}^d)^* = (d^2H(\sigma).\Lambda_2)^\perp \oplus \langle \Lambda_2 \rangle$. We define $P_{\Lambda_2}^\sigma$ to be the projection onto $\langle \Lambda_2 \rangle$ with respect to this decomposition.

4.2. Step k of the construction. In the following, we set $\Lambda = \Lambda_1$, corresponding to step $k = 1$. We now describe the outcome of our decomposition at step k ($k \geq 1$); we will indicate in §4.3 how to go from step k to $k + 1$, for $k \geq 1$.

At step k , we have decomposed $\mu(t, \cdot)$ as a sum

$$\mu(t, \cdot) = \sum_{1 \leq l \leq k} \sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_l} \mu_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, \cdot) + \sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_k} \mu^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot),$$

where the sums run over the *strictly decreasing* sequences of primitive submodules of $(\mathbb{Z}^d)^*$ (of lengths $l \leq k$ in the first term, of length k in the second term). We have

$$\begin{aligned} \mu_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, x, \xi) &= \int_{R_{\Lambda_2}^{\Lambda_1}(\xi) \times \dots \times R_{\Lambda_l}^{\Lambda_{l-1}}(\xi) \times \langle \Lambda_l \rangle} \tilde{\mu}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}(t, x, \xi, d\eta_1, \dots, d\eta_l) \big|_{\mathbb{T}^d \times R_{\Lambda_1}}, \\ \mu^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, x, \xi) &= \int_{R_{\Lambda_2}^{\Lambda_1}(\xi) \times \dots \times R_{\Lambda_k}^{\Lambda_{k-1}}(\xi) \times \mathbb{S}\langle \Lambda_k \rangle} \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, x, \xi, d\eta_1, \dots, d\eta_k) \big|_{\mathbb{T}^d \times R_{\Lambda_1}}. \end{aligned}$$

The distributions $\tilde{\mu}_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}$ have the following properties :

- (1) $\tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}} \in \mathcal{C}(\mathbb{R}, \mathcal{D}'(T^*\mathbb{T}^d \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_{l-1}\rangle \times \langle\Lambda_l\rangle))$ and all its x -Fourier modes are in Λ_l ; with respect to ξ it is supported in I_{Λ_1} ;
- (2) for every $t \in \mathbb{R}$, $\tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}(t, \cdot)$ is invariant under the flows ϕ_s^j ($j = 0, 1, \dots, l-1$) defined by

$$\begin{aligned}\phi_s^0(x, \xi, \eta_1, \dots, \eta_l) &= (x + sdH(\xi), \xi, \eta_1, \dots, \eta_{l-1}, \eta_l); \\ \phi_s^j(x, \xi, \eta_1, \dots, \eta_l) &= (x + sd^2H(\xi) \frac{\eta_j}{|\eta_j|}, \xi, \eta_1, \dots, \eta_l);\end{aligned}$$

- (3) if $\tau_h \ll 1/h$ then for every $t \in \mathbb{R}$, $\tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}(t, \cdot)$ is a positive measure and

$$\tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}(t, \cdot) = \left(\tilde{\phi}_t^l\right)_* \tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}(0, \cdot),$$

where, for $(x, \xi, \eta_1, \dots, \eta_l) \in T^*\mathbb{T}^d \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_{l-1}\rangle \times \langle\Lambda_l\rangle$ we define:

$$\tilde{\phi}_s^l : (x, \xi, \eta_1, \dots, \eta_l) \mapsto (x + sd^2H(\xi)\eta_l, \xi, \eta_1, \dots, \eta_l);$$

- (4) if $\tau_h = 1/h$ then $\int_{\langle\Lambda_l\rangle} \tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}(t, \cdot, d\eta_l)$ is in $\mathcal{C}(\mathbb{R}, \mathcal{M}_+(T^*\mathbb{T}^d \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_{l-1}\rangle))$ and the measure $\int_{(\mathbb{R}^d)^* \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_{l-1}\rangle \times \langle\Lambda_l\rangle} \tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}(t, \cdot, d\xi, d\eta_1, \dots, d\eta_l)$ is an absolutely continuous measure on \mathbb{T}^d . Besides, if $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ has only Fourier modes in Λ_l , then, define $\mathcal{L}(\mathfrak{F}_l)$ the bundle over $(\langle\Lambda_l\rangle/\Lambda_l) \times I_{\Lambda_1} \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_{l-1}\rangle$ formed of elements $(\omega, \sigma, \eta_1, \dots, \eta_{l-1}, Q)$ where $Q \in \mathcal{L}(L_\omega^2(\mathbb{R}^d, \Lambda_l))$, define similarly $\mathcal{K}(\mathfrak{F}_l)$ and $\mathcal{L}^1(\mathfrak{F}_l)$, then

$$\begin{aligned}\int_{T^*\mathbb{T}^d \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_{l-1}\rangle \times \langle\Lambda_l\rangle} a(x, \xi) \tilde{\mu}_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}(t, dx, d\xi, d\eta_1, \dots, d\eta_l) = \\ \int_{((\Lambda_l)/\Lambda_l) \times I_{\Lambda_1} \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_{l-1}\rangle} \text{Tr} \left(a_\sigma \rho_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}(t, d\sigma, d\eta_1, \dots, d\eta_{l-1}) \right),\end{aligned}$$

where $\rho_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}$ is L^∞ in t , a positive section element of $\Gamma(K(\mathfrak{F}_l))'$ and where a_σ is the section of $\mathcal{L}(\mathfrak{F}_l)$ defined by multiplication by $a(\sigma, y)$.

When $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ then $\rho_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}} = M_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}} m_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}$ where

$$m_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}} \in \mathcal{M}_+((\langle\Lambda_l\rangle/\Lambda_l) \times I_{\Lambda_1} \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_{l-1}\rangle),$$

$M_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}$ is a section of $\mathcal{L}^1(\mathfrak{F}_l)$ integrable with respect to $m_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}$. Moreover, $\text{Tr}_{L_\omega^2(\mathbb{R}^d, \Lambda_l)} M_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}} = 1$ and $M_{\Lambda_l}^{\Lambda_1\Lambda_2\cdots\Lambda_{l-1}}$ satisfies a Heisenberg equation $(\text{Heis}_{\Lambda, \omega, \sigma})$ with $\Lambda = \Lambda_l$.

On the other hand $\tilde{\mu}^{\Lambda_1\Lambda_2\cdots\Lambda_k}$ satisfy:

- (1) $\tilde{\mu}^{\Lambda_1\Lambda_2\cdots\Lambda_k}$ is in $L^\infty(\mathbb{R}, \mathcal{M}_+(T^*\mathbb{T}^d \times \mathbb{S}\langle\Lambda_1\rangle \times \cdots \times \mathbb{S}\langle\Lambda_k\rangle))$ and all its x -Fourier modes are in Λ_k ;
- (2) $\tilde{\mu}^{\Lambda_1\Lambda_2\cdots\Lambda_k}$ is invariant by the $k+1$ flows, $\phi_s^0 : (x, \xi, \eta) \mapsto (x + sdH(\xi), \xi, \eta_1, \dots, \eta_k)$, and $\phi_s^l : (x, \xi, \eta_1, \dots, \eta_k) \mapsto (x + sd^2H(\sigma(\xi)) \frac{\eta_l}{|\eta_l|}, \xi, \eta_1, \dots, \eta_k)$ (where $l = 1, \dots, k$).

Finally, we define the space $\mathcal{S}_{\Lambda_k}^k$ which is the class of smooth functions $a(x, \xi, \eta_1, \dots, \eta_k)$ on $T^*\mathbb{T}^d \times \langle \Lambda_1 \rangle \times \dots \times \langle \Lambda_k \rangle$ that are

- (i) smooth and compactly supported in $(x, \xi) \in T^*\mathbb{T}^d$;
- (ii) homogeneous of degree 0 at infinity in each variable η_1, \dots, η_k ;
- (iii) such that their non-vanishing x -Fourier coefficients correspond to frequencies in Λ_k .

4.3. From step k to step $k+1$ ($k \geq 1$). After step k , we leave untouched the term $\sum_{1 \leq l \leq k} \sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_l} \mu_{\Lambda_l}^{\Lambda_1 \Lambda_2 \dots \Lambda_{l-1}}$ and decompose further $\sum_{\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_k} \mu^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$. Using the positivity of $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$, we use the procedure described in Section 2.1 to write

$$(57) \quad \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(\sigma, \cdot) = \sum_{\Lambda_{k+1} \subset \Lambda_k} \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k} \rfloor_{\eta_k \in R_{\Lambda_{k+1}}^k}(\sigma),$$

where the sum runs over all primitive submodules Λ_{k+1} of Λ_k . Moreover, by Proposition 2.1, all the x -Fourier modes of $\tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k} \rfloor_{\eta_k \in R_{\Lambda_{k+1}}^k}(\sigma)$ are in Λ_{k+1} . To generalize the analysis of Section 2.2, we consider test functions in $\mathcal{S}_{\Lambda_{k+1}}^{k+1}$. We let

$$w_{h, R_1, \dots, R_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, x, \xi, \eta_1, \dots, \eta_{k+1}) := \left(1 - \chi\left(\frac{\eta_{k+1}}{R_{k+1}}\right)\right) \\ \times w_{h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, x, \xi, \eta_1, \dots, \eta_k) \otimes \delta_{P_{\Lambda_{k+1}}^\xi(\eta_k)}(\eta_{k+1}),$$

and

$$w_{\Lambda_{k+1}h, R_1, \dots, R_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, x, \xi, \eta_1, \dots, \eta_{k+1}) := \chi\left(\frac{\eta_{k+1}}{R_{k+1}}\right) \\ \times w_{h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, x, \xi, \eta_1, \dots, \eta_k) \otimes \delta_{P_{\Lambda_{k+1}}^\xi(\eta_k)}(\eta_{k+1}).$$

By the Calderón-Vaillancourt theorem, both $w_{\Lambda_{k+1}h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}$ and $w_{h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}$ are bounded in $L^\infty(\mathbb{R}, (\mathcal{S}_{\Lambda_{k+1}}^{k+1})')$. After possibly extracting subsequences, we can take the following limits :

$$\lim_{R_{k+1} \rightarrow +\infty} \dots \lim_{R_1 \rightarrow +\infty} \lim_{h \rightarrow 0} \left\langle w_{h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t), a \right\rangle =: \left\langle \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t), a \right\rangle,$$

and

$$\lim_{R_{k+1} \rightarrow +\infty} \dots \lim_{R_1 \rightarrow +\infty} \lim_{h \rightarrow 0} \left\langle w_{\Lambda_{k+1}h, R_1, \dots, R_k}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t), a \right\rangle =: \left\langle \tilde{\mu}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t), a \right\rangle.$$

Then the properties listed in the preceding subsection are a direct generalisation of Theorems 2.5 and 2.6 (see also [3], Section 4) and of the identity

$$(58) \quad \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot) \rfloor_{\eta_k \in R_{\Lambda_{k+1}}^k}(\sigma) = \int_{\langle \Lambda_{k+1} \rangle} \tilde{\mu}^{\Lambda_1 \Lambda_2 \dots \Lambda_{k+1}}(t, \cdot, d\eta_{k+1}) \rfloor_{\eta_k \in R_{\Lambda_{k+1}}^k}(\sigma) \\ + \int_{\langle \Lambda_{k+1} \rangle} \tilde{\mu}_{\Lambda_{k+1}}^{\Lambda_1 \Lambda_2 \dots \Lambda_k}(t, \cdot, d\eta_{k+1}) \rfloor_{\eta_k \in R_{\Lambda_{k+1}}^k}(\sigma).$$

Remark 4.1. By construction, if $\Lambda_{k+1} = \{0\}$, we have $\tilde{\mu}^{\Lambda_1\Lambda_2\ldots\Lambda_{k+1}} = 0$, and the induction stops. Similarly to Remark 2.8, one can also see that if $\text{rk } \Lambda_{k+1} = 1$, the invariance properties of $\tilde{\mu}^{\Lambda_1\Lambda_2\ldots\Lambda_{k+1}}$ imply that it is constant in x .

Remark 4.2. Note that in the preceding definition of k -microlocal Wigner transform for $k \geq 1$, we did not use a parameter δ tending to 0 as we did when $k = 0$ in order to isolate the part of the limiting measures supported above $R_{\Lambda_{k+1}}^{\Lambda_k}(\sigma)$. This comes directly from the restrictions made in (57) and (58).

4.4. Proof of Theorem 1.10. This iterative procedure allows to decompose μ along decreasing sequences of submodules. In particular, when $\tau_h \sim 1/h$, it implies Theorem 1.10. Indeed, to end the proof of Theorem 1.10, we let after the final step of the induction

$$\begin{aligned} \mu_{\Lambda}^{\text{final}}(t, \cdot) &= \sum_{0 \leq k \leq d} \sum_{\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_k \supset \Lambda} \mu_{\Lambda}^{\Lambda_1\Lambda_2\ldots\Lambda_k}(t, \cdot) \\ &= \sum_{0 \leq k \leq d} \sum_{\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_k \supset \Lambda} \int_{R_{\Lambda_2}^{\Lambda_1}(\xi) \times \cdots \times R_{\Lambda}^{\Lambda_k}(\xi) \times \langle \Lambda \rangle} \tilde{\mu}_{\Lambda}^{\Lambda_1\Lambda_2\ldots\Lambda_k}(t, \cdot, d\eta_1, \ldots, d\eta_k) \rfloor_{\mathbb{T}^d \times R_{\Lambda_1}}, \end{aligned}$$

where $\Lambda_1, \ldots, \Lambda_k$ run over the set of strictly decreasing sequences of submodules ending with Λ . We know that $\mu_{\Lambda}^{\Lambda_1\Lambda_2\ldots\Lambda_k}$ is supported on $\{\xi \in I_{\Lambda_1}\}$, and since $\Lambda \subset \Lambda_1$ we have $I_{\Lambda_1} \subset I_{\Lambda}$.

We also let

$$\rho_{\Lambda}^{\text{final}}(t, \omega, \sigma) = \sum_{0 \leq k \leq d} \sum_{\Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_k \supset \Lambda} \int_{R_{\Lambda_2}^{\Lambda_1}(\xi) \times \cdots \times R_{\Lambda}^{\Lambda_k}(\xi)} \tilde{\rho}_{\Lambda}^{\Lambda_1\Lambda_2\ldots\Lambda_k}(t, \omega, \sigma, d\eta_1, \ldots, d\eta_k) \rfloor_{\sigma \in R_{\Lambda_1}},$$

where the $\tilde{\rho}_{\Lambda}^{\Lambda_1\Lambda_2\ldots\Lambda_k}$ are the operator-valued measures appearing in §4.2.

Remark 4.3. It is clear from this construction that $\rho_{\Lambda}^{\text{final}}$ and $\mu_{\Lambda}^{\text{final}}(t, \cdot)$ can only charge those $\sigma \in (\mathbb{R}^d)^*$ with $\Lambda \subseteq dH(\sigma)^{\perp}$. Moreover, if $\mathbf{V}_h(t) = \text{Op}_h(V(t, \cdot))$ the measure $\rho_{\Lambda}^{\text{final}}$ admits a decomposition $\rho_{\Lambda}^{\text{final}} = N_{\Lambda} \bar{\mu}_{\Lambda}$ where $\bar{\mu}_{\Lambda}$ is a measure that does not depend on t and $N_{\Lambda}(\cdot, \omega, \sigma)$ is a family of positive, trace-class operators on $L_{\omega}^2(\mathbb{R}^d, \Lambda)$ with $\text{Tr } N_{\Lambda} \equiv 1$, satisfying the propagation law $(\text{Heis}_{\Lambda, \omega, \sigma})$.

As already mentioned, Theorem 1.10 implies Theorem 1.3 in the case $\tau_h \sim 1/h$. The proof of Theorem 1.3 in the case $\tau_h \ll 1/h$ is discussed in Section 5 and in the case $\tau_h \gg 1/h$, in Section 6.

4.5. Sufficient assumptions. In the induction, we used the fact that

$$(59) \quad \langle \Lambda_k \rangle = \langle \Lambda_{k+1} \rangle \oplus (d^2 H(\xi) \cdot \langle \Lambda_{k+1} \rangle^{\perp} \cap \langle \Lambda_k \rangle) \text{ for all } k.$$

Definiteness of the Hessian $d^2 H(\xi)$ is certainly a sufficient assumption for this, but we see that we actually need less if we note we are not using this property for arbitrary Λ_k , but only for the ones arising in the construction (remember for instance that for $k = 1$ we only need (59) for $\Lambda_1 = dH(\xi)^{\perp} \cap \mathbb{Z}^d$).

A careful analysis of the proof shows that a sufficient set of assumptions is the following :

Assumption 4.4. *For every integer k , for all $\xi, \eta_1, \dots, \eta_k \in (\mathbb{R}^d)^*$, for every strictly decreasing sequence of primitive submodules $\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_k \supset \{0\}$ such that:*

$$\Lambda_1 = dH(\xi)^\perp \cap \mathbb{Z}^d, \Lambda_2 = (d^2H(\xi) \cdot \eta_1)^\perp \cap \Lambda_1, \dots, \Lambda_k = (d^2H(\xi) \cdot \eta_{k-1})^\perp \cap \Lambda_{k-1},$$

and

$\eta_1 \in \langle \Lambda_1 \rangle \setminus \{0\}, \eta_2 \in (d^2H(\xi) \cdot \eta_1)^\perp \setminus (d^2H(\xi) \cdot \Lambda_1)^\perp, \eta_k \in (d^2H(\xi) \cdot \eta_{k-1})^\perp \setminus (d^2H(\xi) \cdot \Lambda_{k-1})^\perp$
then $d^2H(\xi) \cdot \eta_k \notin \Lambda_k^\perp$.

We leave it to the reader to check that Assumptions 4.4 implies (59) and thus is a sufficient assumption for all our results.

In dimension $d = 2$, Assumptions 4.4 is implied by isoenergetic non-degeneracy (whereas we saw that it is no longer the case for $d \geq 3$). In dimension 2, what happens is that, either $dH(\xi)$ is a vector with rationally independent entries (in which case $\Lambda_1 = \{0\}$ and the conditions of Assumptions 4.4 are empty), or $dH(\xi)$ is a non zero vector with rationally dependent entries : in this case (and this is very special to dimension 2), Λ_1^\perp is one-dimensional and coincides with $\mathbb{R}dH(\xi)$. Thus Assumptions 4.4 just says that

$$dH(\xi) \cdot \eta_1 = 0, \eta_1 \neq 0 \implies dH^2(\xi) \cdot \eta_1 \notin \mathbb{R}dH(\xi)$$

which is isoenergetic non-degeneracy. Remark that $dH(\xi) = 0$ is forbidden by isoenergetic non-degeneracy.

Note, finally, that isoenergetic non-degeneracy is only a local condition at ξ (since it involves only $dH(\xi), d^2H(\xi)$) whereas condition Assumptions 4.4 contains some global features, namely the relations between $dH(\xi), d^2H(\xi)$ and the ring \mathbb{Z}^d , which is the homology group of \mathbb{T}^d .

5. SOME EXAMPLES OF SINGULAR CONCENTRATION

In Subsection 5.1, assuming $\mathbf{V}_h(t) = 0$, we present some examples of singular concentrations for the scales $\tau_h \ll h$ and, in that manner, we conclude the proof of Theorem 1.3 by proving the only remaining point (1). Then the two other subsections are devoted to the analysis of other cases of singular concentration which arise when the assumptions of Theorem 1.3 are not satisfied.

5.1. Singular concentration for time scales $\tau_h \ll 1/h$. Assume $\mathbf{V}_h(t) = 0$ and consider $\rho \in \mathcal{S}(\mathbb{R}^d)$ with $\|\rho\|_{L^2(\mathbb{R}^d)} = 1$ and such that the Fourier transform $\hat{\rho}$ is compactly supported. Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and (ε_h) a sequence of positive real numbers that tends to zero as $h \rightarrow 0^+$. Form the wave-packet:

$$(60) \quad v_h(x) := \frac{1}{(\varepsilon_h)^{d/2}} \rho\left(\frac{x - x_0}{\varepsilon_h}\right) e^{i\frac{\xi_0}{h} \cdot x}.$$

Define $u_h := \mathbf{P}v_h$, where \mathbf{P} denotes the periodization operator $\mathbf{P}v(x) := \sum_{k \in \mathbb{Z}^d} v(x + 2\pi k)$. Since ρ is rapidly decreasing, we have $\|u_h\|_{L^2(\mathbb{T}^d)} \xrightarrow{h \rightarrow 0} 1$. The family (u_h) is h -oscillatory if $\varepsilon_h \gg h$.

Theorem 1.3(1) is a consequence of our next result.

Proposition 5.1. *Let (τ_h) be such that $\lim_{h \rightarrow 0^+} h\tau_h = 0$; suppose that $\varepsilon_h \gg h\tau_h$. Then the Wigner distributions of the solutions $S_h^{\tau_h t} u_h$ converge weakly-* in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$ to $\mu_{(x_0, \xi_0)}$, defined by:*

$$(61) \quad \int_{T^*\mathbb{T}^d} a(x, \xi) \mu_{(x_0, \xi_0)}(dx, d\xi) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a(x_0 + tdH(\xi_0), \xi_0) dt, \quad \forall a \in \mathcal{C}_c(T^*\mathbb{T}^d).$$

We call the measures μ_{x_0, ξ_0} “uniform orbit measures” for ϕ_s (their definition and existence as a limit is specific to translation flows on the torus). They are H -invariant and the convex hull of the set of uniform orbit measures is dense in the set of H -invariant measures. Considering initial data that are linear combinations of wave packets of the form (60), we see that the convex hull of uniform orbit measures is contained in $\widetilde{\mathcal{M}}(\tau)$, and since the latter is closed, it contains all measures invariant by ϕ_s as stated in Theorem 1.3(1).

Proof. Start noticing that the sequence of initial conditions (u_h) possesses the unique semiclassical measure $\mu_0 = \delta_{x_0} \otimes \delta_{\xi_0}$. Using property (4) in the appendix, we deduce that the image $\bar{\mu}$ of $\mu(t, \cdot)$ by the projection from $\mathbb{T}^d \times \mathbb{R}^d$ onto \mathbb{R}^d satisfies:

$$\bar{\mu} = \sum_{\Lambda \in \mathcal{L}} \bar{\mu}|_{R_\Lambda} = \delta_{\xi_0}.$$

Since the sets R_Λ form a partition of \mathbb{R}^d , we conclude that $\bar{\mu}|_{R_\Lambda} = 0$ unless $\Lambda = \Lambda_{\xi_0}$ and therefore $\mu = \mu|_{\mathbb{T}^d \times R_{\Lambda_{\xi_0}}}$. Therefore, in order to characterize μ it suffices to test it against symbols with Fourier coefficients in Λ_{ξ_0} . Let $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ be such a symbol; we can restrict our attention to the case where a is a trigonometric polynomial in x . Let $\varphi \in L^1(\mathbb{R})$. Recall that the Wigner distributions $w_h(t)$ of $S_h^{\tau_h t} u_h$ satisfy

$$\int_{\mathbb{R}} \varphi(t) \langle w_h(t), a \rangle dt = \int_{\mathbb{R}} \varphi(t) \langle w_h(0), a \circ \phi_{\tau_h t} \rangle dt + o(1);$$

moreover the Poisson summation formula ensures that the Fourier coefficients of u_h are given by:

$$\widehat{u_h}(k) = \frac{(\varepsilon_h)^{d/2}}{(2\pi)^{d/2}} \widehat{\rho} \left(\frac{\varepsilon_h}{h} (hk - \xi_0) \right) e^{-i(k - \xi_0/h) \cdot x_0}.$$

Combining this with the explicit formula (75) for the Wigner distribution presented in the appendix we get:

$$(62) \quad \int_{\mathbb{R}} \varphi(t) \langle w_h(t), a \rangle dt = \frac{(\varepsilon_h)^d}{(2\pi)^{3d/2}} \sum_{k-j \in \Lambda_{\xi_0}} \widehat{\varphi} \left(\tau_h dH \left(h \frac{k+j}{2} \right) \cdot (k-j) \right) \widehat{a}_{j-k} \left(h \frac{k+j}{2} \right) \\ \widehat{\rho} \left(\frac{\varepsilon_h}{h} (hk - \xi_0) \right) \overline{\widehat{\rho} \left(\frac{\varepsilon_h}{h} (hj - \xi_0) \right)} e^{-i(k-j) \cdot x_0} + o(1).$$

Now, since $k - j \in \Lambda_{\xi_0}$ we can write:

$$\begin{aligned} \left| dH \left(h \frac{k+j}{2} \right) \cdot (k-j) \right| &= \left| \left[dH \left(h \frac{k+j}{2} \right) - dH(\xi_0) \right] \cdot (k-j) \right| \\ &\leq C \left| h \frac{k+j}{2} - \xi_0 \right| |k-j|. \end{aligned}$$

By hypothesis, both $\widehat{\rho}$ and $k \mapsto \widehat{a}_k(\xi)$ are compactly supported, and hence the sum (62) only involves terms satisfying:

$$\frac{\varepsilon_h}{h} \left| h \frac{k}{2} - \xi_0 \right| \leq R, \quad \frac{\varepsilon_h}{h} \left| h \frac{j}{2} - \xi_0 \right| \leq R \quad \text{and} \quad |j-k| \leq R$$

for some fixed R . This in turn implies

$$\left| \tau_h dH \left(h \frac{k+j}{2} \right) \cdot (k-j) \right| \leq CR^2 \frac{\tau_h h}{\varepsilon_h}.$$

This shows that the limit of (62) as $h \rightarrow 0^+$ coincides with that of:

$$\begin{aligned} &\frac{(\varepsilon_h)^d}{(2\pi)^{3d/2}} \sum_{k-j \in \Lambda_{\xi_0}} \widehat{\varphi}(0) a_{j-k} \left(h \frac{k+j}{2} \right) \widehat{\rho} \left(\frac{\varepsilon_h}{h} (hk - \xi_0) \right) \overline{\widehat{\rho} \left(\frac{\varepsilon_h}{h} (hj - \xi_0) \right)} e^{-i(k-j) \cdot x_0} \\ &= \widehat{\varphi}(0) \langle w_h(0), a \rangle, \end{aligned}$$

which is precisely:

$$\widehat{\varphi}(0) a(x_0, \xi_0) = \widehat{\varphi}(0) \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a(x_0 + tdH(\xi_0), \xi_0) dt,$$

since a has only Fourier modes in Λ_{ξ_0} . □

We next present a slight modification of the previous example in order to illustrate the two-microlocal nature of the elements of $\widetilde{\mathcal{M}}(\tau)$. Define now, for $\eta_0 \in \mathbb{R}^d$:

$$u_h(x) = \mathbf{P} \left[v_h(x) e^{i\eta_0/(h\tau_h)} \right],$$

where v_h was defined in (60).

Proposition 5.2. *Suppose that $\lim_{h \rightarrow 0^+} h\tau_h = 0$ and $\varepsilon_h \gg h\tau_h$. Suppose moreover that $d^2H(\xi_0)$ is definite and that $\eta_0 \in \langle \Lambda_{\xi_0} \rangle$. Then the Wigner distributions of $S_h^{\tau_h t} u_h$ converge weakly-* in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$ to the measure:*

$$\mu(t, \cdot) = \mu_{(x_0 + td^2H(\xi_0)\eta_0, \xi_0)}, \quad t \in \mathbb{R},$$

where $\mu_{(x_0, \xi_0)}$ is the uniform orbit measure defined in (61).

Proof. The same argument we used in the proof of Proposition 5.1 gives $\mu = \mu|_{\mathbb{T}^d \times R_{\Lambda_{\xi_0}}}$. We claim that $w_{I_{\Lambda_{\xi_0}}, h, R}(0)$ converges to the measure:

$$(63) \quad \widetilde{\mu}_{\Lambda_{\xi_0}}(0, x, \xi, \eta) = \mu_{(x_0, \xi_0)}(x, \xi) \delta_{\eta_0}(\eta).$$

Assume this is the case, Theorem 2.6 (4) implies that:

$$\tilde{\mu}_{\Lambda_{\xi_0}}(t, x, \xi, \eta) = \mu_{(x_0+td^2H(\xi_0)\eta_0, \xi_0)}(x, \xi) \delta_{\eta_0}(\eta), \quad \forall t \in \mathbb{R},$$

and, since $\tilde{\mu}_{\Lambda_{\xi_0}}(t, \cdot)$ are probability measures, it follows from Proposition 2.3 that $\tilde{\mu}^{\Lambda_{\xi_0}} = 0$ and :

$$\mu(t, \cdot) = \int_{\langle \Lambda_{\xi_0} \rangle} \tilde{\mu}_{\Lambda_{\xi_0}}(t, \cdot, d\eta) = \mu_{(x_0+td^2H(\xi_0)\eta_0, \xi_0)}.$$

Let us now prove the claim (63). Set

$$\tilde{u}_h(x) = v_h(x) e^{i\eta_0/(h\tau_h)}.$$

Consider $h_0 > 0$ and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\chi \tilde{u}_h = \tilde{u}_h$ for all $h \in (0, h_0)$ and $\mathbf{P}\chi^2 \equiv 1$. We now take $a \in \mathcal{S}_\Lambda^1$ and denote by \tilde{a} the smooth compactly supported function defined on \mathbb{R}^d by $\tilde{a} = \chi^2 a$. Using the fact that the two-scale quantization admits the gain $h\tau_h$ (in view of (25)),

$$\begin{aligned} \langle u_h, \text{Op}_h^{\Lambda_{\xi_0}}(a)u_h \rangle_{L^2(\mathbb{T}^d)} &= \langle u_h, \text{Op}_h^{\Lambda_{\xi_0}}(\tilde{a})u_h \rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \tilde{u}_h, \text{Op}_h^{\Lambda_{\xi_0}}(a)\tilde{u}_h \rangle_{L^2(\mathbb{R}^d)} + O(h\tau_h). \end{aligned}$$

Therefore, it is possible to lift the computation of the limit of $w_{I_{\Lambda_{\xi_0}}, h, R}(0)$ to $T^*\mathbb{R}^d \times \langle \Lambda_{\xi_0} \rangle$ and, in consequence, replace sums by integrals. A direct computation gives:

$$\begin{aligned} \langle \tilde{u}_h, \text{Op}_h^{\Lambda_{\xi_0}}(a)\tilde{u}_h \rangle_{L^2(\mathbb{R}^d)} &= (2\pi)^{-d} \int_{\mathbb{R}^{3d}} e^{i\xi \cdot (x-y)} \bar{\rho}(x) \rho(y) \\ &\quad \times a \left(x_0 + \varepsilon_h \frac{x+y}{2}, \xi_0 + \frac{1}{\tau_h} \eta_0 + \frac{h}{\varepsilon_h} \xi, \tau_h \eta \left(\xi_0 + \frac{1}{\tau_h} \eta_0 + \frac{h}{\varepsilon_h} \xi \right) \right) dx dy d\xi. \end{aligned}$$

Note that if $F(\xi) = (\sigma, \eta)$, then

$$\forall k \in \Lambda, \quad F(\xi + k) = (\sigma, \eta + k) = F(\xi) + (0, k),$$

which implies that $dF(\xi)k = (0, k)$ and $d\eta(\xi)k = k$ for all $k \in \Lambda_{\xi_0}$. We deduce $d\eta(\xi_0)\eta_0 = \eta_0$ since $\eta_0 \in \langle \Lambda_{\xi_0} \rangle$ and, in view of $\eta(\xi_0) = 0$, a Taylor expansion of $\eta(\xi)$ around ξ_0 gives

$$\tau_h \eta \left(\xi_0 + \frac{1}{\tau_h} \eta_0 + \frac{h}{\varepsilon_h} \xi \right) = \eta_0 + o(1).$$

Therefore, as h goes to 0,

$$\langle \tilde{u}_h, \text{Op}_h^{\Lambda_{\xi_0}}(a)\tilde{u}_h \rangle \rightarrow a(x_0, \xi_0, \eta_0) = \langle \tilde{\mu}_{\Lambda_{\xi_0}}, a \rangle.$$

□

5.2. Singular concentration for Hamiltonians with critical points. We next show by a quasimode construction that for Hamiltonians having a degenerate critical point (of order $k > 2$) and for time scales $\tau_h \ll 1/h^{k-1}$, the set $\widetilde{\mathcal{M}}(\tau)$ always contains singular measures.

Suppose $\xi_0 \in \mathbb{R}^d$ is such that:

$$dH(\xi_0), d^2H(\xi_0), \dots, d^{k-1}H(\xi_0) \quad \text{vanish identically.}$$

The Hamiltonian $H(\xi) = |\xi|^k$ (k an even integer – corresponding to the operator $(-\Delta)^{\frac{k}{2}}$) provides such an example (with $\xi_0 = 0$). Let $u_h = \mathbf{P}v_h$, where v_h is defined in (60). If $\varepsilon_h \gg h$ it is not hard to see that

$$\|H(hD_x)u_h - H(\xi_0)u_h\|_{L^2(\mathbb{T}^d)} = O\left(h^k/(\varepsilon_h)^k\right).$$

Therefore,

$$\left\|S_h^t u_h - e^{-i\frac{t}{h}H(\xi_0)}u_h\right\|_{L^2(\mathbb{T}^d)} = t O\left(h^{k-1}/(\varepsilon_h)^k\right),$$

and, it follows that, for compactly supported $\varphi \in L^1(\mathbb{R})$ and $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$,

$$\int_{\mathbb{R}} \varphi(t) \langle w_h(t), a \rangle dt = \int_{\mathbb{R}} \varphi(t) \langle u_h, \text{Op}_h(a)u_h \rangle_{L^2(\mathbb{T}^d)} dt + O\left(\tau_h h^{k-1}/(\varepsilon_h)^k\right).$$

Choosing (ε_h) tending to zero and such that $\varepsilon_h \gg (\tau_h h^{k-1})^{1/k}$, the latter quantity converges to $a(x_0, \xi_0) \|\varphi\|_{L^1(\mathbb{R})}$ as $h \rightarrow 0^+$. In other words,

$$dt \otimes \delta_{x_0} \otimes \delta_{\xi_0} \in \widetilde{\mathcal{M}}(\tau),$$

whence $dt \otimes \delta_{x_0} \in \mathcal{M}(\tau)$.

In the special case of $H(\xi) = |\xi|^k$ (k an even integer), we know that the threshold τ_h^H is precisely h^{1-k} . From the discussion of §6 and previously known results about eigenfunctions of the laplacian, we know that the elements of $\mathcal{M}(\tau)$ are absolutely continuous for scales $\tau_h \gg 1/h^{k-1}$. In the case of $\tau_h = 1/h^{k-1}$, one can still show that elements of $\mathcal{M}(\tau)$ are absolutely continuous. This requires some extra work which consists in checking that all our proofs still work in this case for $\tau_h = 1/h^{k-1}$ and ξ in a neighbourhood of $\xi_0 = 0$, replacing the Hessian $d^2H(\xi_0)$ by $d^kH(\xi_0)$, and the assumption that the Hessian is definite by the assumption that $[d^kH(\xi_0).\xi^k = 0 \implies \xi = 0]$.

In the general case of a Hamiltonian having a degenerate critical point, the existence of such a threshold, and its explicit determination, is by no means obvious.

5.3. The effect of the presence of a subprincipal symbol of lower order in h .

Here we present some remarks concerning how the preceding results may change when the Hamiltonian $H(hD_x)$ is perturbed by a potential $h^\beta \mathbf{V}_h(t)$ with $\beta \in (0, 2)$ and $\mathbf{V}_h(t)$ is a multiplication operator by some smooth function $V(t, x)$. In this case, it is possible to find potentials $V(t, x)$ for which Theorem 1.3(2) fails, *i.e.* such that there exists $\mu \in \widetilde{\mathcal{M}}(1/h)$, the projection of which on x is not absolutely continuous with respect to $dt dx$. The following example has been communicated to us by Jared Wunsch. On the 2-dimensional

torus, take $H(\xi) = |\xi|^2$ and $V(x_1, x_2) := W(x_2)$ such that $W(x_2) = (x_2)^2/2$ in the set $\{|x_2| < 1/2\}$. Take $\varepsilon \in (0, 1)$ and

$$u_h(x, y) := \frac{1}{\pi^{1/4} h^{\varepsilon/4}} e^{i \frac{x_1}{h}} e^{-\frac{(x_2)^2}{2h^\varepsilon}} \chi(y),$$

where χ is a smooth function that is equal to one in $\{|x_2| < 1/4\}$ and identically equal to 0 in $\{|x_2| > 1/2\}$. One checks that

$$(-h^2 \Delta + h^{2(1-\varepsilon)} V - 1) u_h = h^{2-\varepsilon} u_h + O(h^\infty).$$

It follows that for $\varphi \in L^1(\mathbb{R})$ and $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2)$,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \varphi(t) \left\langle S_{2(1-\varepsilon), h}^{t/h} u_h, \text{Op}_h(a) S_{2(1-\varepsilon), h}^{t/h} u_h \right\rangle_{L^2(\mathbb{T}^2)} dt \\ = \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \varphi(t) \langle u_h, \text{Op}_h(a) u_h \rangle_{L^2(\mathbb{T}^2)} dt \\ = \left(\int_{\mathbb{R}} \varphi(t) dt \right) \lim_{h \rightarrow 0^+} \langle u_h, \text{Op}_h(a) u_h \rangle_{L^2(\mathbb{T}^2)} \\ = \left(\int_{\mathbb{R}} \varphi(t) dt \right) \int_{T^*\mathbb{T}^2} a(x, \xi) \mu(dx, d\xi), \end{aligned}$$

and it is not hard to see that μ is concentrated on $\{x_2 = 0, \xi_1 = 1, \xi_2 = 0\}$. In particular the image of μ by the projection to \mathbb{T}^2 is supported on $\{x_2 = 0\}$.

6. HIERARCHIES OF TIME SCALES

Here we prove the results announced in §1.5 of the introduction. These results make explicit the relation between the sets $\widetilde{\mathcal{M}}(\tau)$ as the time scale (τ_h) varies.

Proposition 6.1. *Let (τ_h) and (σ_h) be time scales tending to infinity as $h \rightarrow 0^+$ such that $\lim_{h \rightarrow 0^+} \sigma_h/\tau_h = 0$. Then for every $\mu \in \widetilde{\mathcal{M}}(\tau)$ and almost every $t \in \mathbb{R}$ there exist $\mu^t \in \text{Conv } \widetilde{\mathcal{M}}(\sigma)$ such that*

$$(64) \quad \mu(t, \cdot) = \int_0^1 \mu^t(s, \cdot) ds.$$

Before presenting the proof of this result, we shall need two auxiliary lemmas.

Lemma 6.2. *Let (σ_h) be a time scale tending to infinity as $h \rightarrow 0^+$. Let $(v_h^{(n)})_{h>0, n \in \mathbb{N}}$ be a normalised family in $L^2(\mathbb{T}^d)$ and define:*

$$w_h^{(n)}(t, \cdot) := w_{S_h^{\sigma_h t} v_h^{(n)}}^h.$$

Let $c_h^{(n)} \geq 0$, $n \in \mathbb{N}$, be such that $\sum_{n \in \mathbb{N}} c_h^{(n)} = 1$. Then, every weak- accumulation point in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$ of*

$$(65) \quad \sum_{n \in I_h} c_h^{(n)} w_h^{(n)}(t, \cdot)$$

belongs to $\text{Conv } \widetilde{\mathcal{M}}(\sigma)$.

Proof. Suppose (65) possesses an accumulation point $\tilde{\mu} \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$ that does not belong to $\text{Conv } \widetilde{\mathcal{M}}(\sigma)$. By the Hahn-Banach theorem applied to the compact convex sets $\{\tilde{\mu}\}$ and $\text{Conv } \widetilde{\mathcal{M}}(\sigma)$ we can ensure the existence of $\varepsilon > 0$, $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ and $\theta \in L^1(\mathbb{R})$ such that:

$$\int_{\mathbb{R}} \theta(t) \langle \tilde{\mu}(t, \cdot), a \rangle dt < -\varepsilon < 0,$$

and,

$$(66) \quad \int_{\mathbb{R}} \theta(t) \langle \mu(t, \cdot), a \rangle dt \geq -\frac{\varepsilon}{3}, \quad \forall \mu \in \text{Conv } \widetilde{\mathcal{M}}(\sigma).$$

Suppose that $\tilde{\mu}$ is attained through a sequence (h_k) tending to zero. For $k > k_0$ big enough,

$$\int_{\mathbb{R}} \theta(t) \sum_{n \in I_{h_k}} c_{h_k}^{(n)} \langle w_{h_k}^{(n)}(t, \cdot), a \rangle dt \leq -\frac{3}{2}\varepsilon,$$

which implies that there exists $n_k \in \mathbb{N}$ such that:

$$(67) \quad \int_{\mathbb{R}} \theta(t) \langle w_{h_k}^{(n_k)}(t, \cdot), a \rangle dt \leq -\frac{3}{2}\varepsilon.$$

Therefore, every accumulation point of $(w_{h_k}^{(n_k)})$ also satisfies (67) which contradicts (66). \square

Lemma 6.3. *Let τ , σ and μ be as in Proposition 6.1. For every $\alpha < \beta$ there exists $\mu_{\alpha, \beta} \in \text{Conv } \widetilde{\mathcal{M}}(\sigma)$ such that*

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mu(t, \cdot) dt = \int_0^1 \mu_{\alpha, \beta}(t, \cdot) dt.$$

Proof. Let $\mu \in \widetilde{\mathcal{M}}(\tau)$. Then there exists an h -oscillating, normalised sequence (u_h) such that, for every $\theta \in L^1(\mathbb{R})$ and every $a \in C_c^\infty(T^*\mathbb{T}^d)$:

$$\lim_{h \rightarrow 0^+} \int_{\mathbb{R}} \theta(t) \langle S_h^{\tau_h t} u_h, \text{Op}_h(a) S_h^{\tau_h t} u_h \rangle dt = \int_{\mathbb{R}} \theta(t) \langle \mu(t, \cdot), a \rangle dt.$$

Write $N_h := \tau_h / \sigma_h$; by hypothesis $N_h \rightarrow \infty$ as $h \rightarrow 0^+$. Let $\alpha < \beta$, define $L := \beta - \alpha$ and put:

$$\delta_h := \frac{LN_h}{\lfloor LN_h \rfloor}, \quad t_n^h := \alpha N_h + n \delta_h,$$

where $\lfloor LN_h \rfloor$ is the integer part of LN_h . Then,

$$\begin{aligned} \frac{1}{L} \int_{\alpha}^{\beta} \langle S_h^{\tau_h t} u_h, \text{Op}_h(a) S_h^{\tau_h t} u_h \rangle_{L^2(\mathbb{T}^d)} dt &= \frac{1}{LN_h} \int_{\alpha N_h}^{\beta N_h} \langle S_h^{\sigma_h t} u_h, \text{Op}_h(a) S_h^{\sigma_h t} u_h \rangle_{L^2(\mathbb{T}^d)} dt \\ &= \frac{1}{LN_h} \sum_{n=1}^{\lfloor LN_h \rfloor} \int_{t_{n-1}^h}^{t_n^h} \langle S_h^{\sigma_h t} u_h, \text{Op}_h(a) S_h^{\sigma_h t} u_h \rangle_{L^2(\mathbb{T}^d)} dt \\ &= \frac{1}{LN_h} \sum_{n=1}^{\lfloor LN_h \rfloor} \int_0^{\delta_h} \langle S_h^{\sigma_h t} v_h^{(n)}, \text{Op}_h(a) S_h^{\sigma_h t} v_h^{(n)} \rangle_{L^2(\mathbb{T}^d)} dt, \end{aligned}$$

where the functions $v_h^{(n)} := S_h^{\sigma_h t_n^h} u_h$ form, for each $n \in \mathbb{Z}$, a normalised sequence indexed by $h > 0$. The result then follows by Lemma 6.2 and using the fact that $\delta_h \rightarrow 1$ as $h \rightarrow 0^+$. \square

Proof of Proposition 6.1. Let $\mu \in \widetilde{\mathcal{M}}(\tau)$; an application of the Lebesgue differentiation theorem gives the existence of a countable dense set $S \subset \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ and a set $N \subset \mathbb{R}$ of measure zero such that, for $a \in S$ and $t \in \mathbb{R} \setminus N$,

$$(68) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \int_{T^*\mathbb{T}^d} a(x, \xi) \mu(s, dx, d\xi) ds = \int_{T^*\mathbb{T}^d} a(x, \xi) \mu(t, dx, d\xi).$$

Fix $t \in \mathbb{R} \setminus N$; then, for any $\varepsilon > 0$ there exist $\mu_\varepsilon^t \in \text{Conv } \widetilde{\mathcal{M}}(\sigma)$ such that, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$,

$$(69) \quad \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \int_{T^*\mathbb{T}^d} a(x, \xi) \mu(s, dx, d\xi) ds = \int_0^1 \int_{T^*\mathbb{T}^d} a(x, \xi) \mu_\varepsilon^t(s, dx, d\xi) ds.$$

Note that $\text{Conv } \widetilde{\mathcal{M}}(\sigma)$ is sequentially compact for the weak-* topology, therefore, there exist a sequence (ε_n) tending to zero and a measure $\mu^t \in \text{Conv } \widetilde{\mathcal{M}}(\sigma)$ such that $\mu_{\varepsilon_n}^t$ converges weakly-* to μ^t . Identities (68) and (69) ensure that $\mu(t, \cdot) = \int_0^1 \mu^t(s, \cdot) ds$. \square

Remark 6.4. Projecting on x in identity (64) we deduce that given $\nu \in \mathcal{M}(\tau)$ there exist $\nu^t \in \mathcal{M}(\sigma)$ such that:

$$\nu(t, \cdot) = \int_0^1 \nu^t(s, \cdot) ds.$$

This, together with the fact that elements of $\mathcal{M}(1/h)$ are absolutely continuous imply the conclusion of Theorem 1.3(2) when $\tau_h \gg 1/h$.

We now assume that $\mathbf{V}_h(t) = 0$. Denote by $\widetilde{\mathcal{M}}(\infty)$ the set of weak-* limit points of sequences of Wigner distributions (w_{u_h}) corresponding to sequences (u_h) consisting of normalised eigenfunctions of $H(hD_x)$. We now focus on a family of time scales τ for which the structure of $\widetilde{\mathcal{M}}(\tau)$ can be described in terms of the closed convex hull of $\widetilde{\mathcal{M}}(\infty)$. Given a measurable subset $O \subseteq \mathbb{R}^d$, we define:

$$\tau_h^H(O) := h \sup \{ |H(hk) - H(hj)|^{-1} : H(hk) \neq H(hj), hk, hj \in h\mathbb{Z}^d \cap O \}.$$

Note that the scale τ_h^H defined in the introduction coincides with $\tau_h^H(\mathbb{R}^d)$. The following holds.

Proposition 6.5. *Let $O \subseteq \mathbb{R}^d$ be an open set such that $\tau_h^H(O)$ tends to infinity as $h \rightarrow 0^+$. Suppose (τ_h) is a time scale such that $\lim_{h \rightarrow 0^+} \tau_h^H(O)/\tau_h = 0$. If $V = 0$ and if $\mu \in \widetilde{\mathcal{M}}(\tau)$ is obtained through a sequence whose semiclassical measure satisfies $\mu_0(\mathbb{T}^d \times (\mathbb{R}^d \setminus O)) = 0$ then $\mu \in \text{Conv } \widetilde{\mathcal{M}}(\infty)$.*

Proof. Since the Fourier coefficient of $S_h^{\tau_h t} u_h$ are $e^{-it \frac{\tau_h}{h} H(hk)} \widehat{u_h}(k)$ and in view of (74) and (75), we can write for $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ and $\theta \in L^1(\mathbb{R})$, we write:

$$\begin{aligned} \int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle dt &= \int_{\mathbb{R}} \theta(t) \langle u_h, S_h^{\tau_h t*} \text{Op}_h(a) S_h^{\tau_h t} u_h \rangle_{L^2(\mathbb{T}^d)} dt \\ &= \frac{1}{(2\pi)^{d/2}} \sum_{k,j \in \mathbb{Z}^d} \widehat{u_h}(k) \overline{\widehat{u_h}(j)} \widehat{a}_{j-k} \left(\frac{h}{2}(k+j) \right) \int_{\mathbb{R}} \theta(t) e^{-it \frac{\tau_h}{h} (H(hk) - H(hj))} dt \\ &= \frac{1}{(2\pi)^{d/2}} \sum_{h,j \in \mathbb{Z}^d} \widehat{\theta} \left(\tau_h \frac{H(hk) - H(hj)}{h} \right) \widehat{u_h}(k) \overline{\widehat{u_h}(j)} \widehat{a}_{j-k} \left(\frac{h}{2}(k+j) \right). \end{aligned}$$

Our assumptions on the semiclassical measure of the initial data implies that, for a.e. $t \in \mathbb{R}$:

$$\mu(t, \mathbb{T}^d \times (\mathbb{R}^d \setminus O)) = 0.$$

Suppose that μ is obtained through the normalised sequence (u_h) . Suppose that $a \in \mathcal{C}_c^\infty(\mathbb{T}^d \times O)$ and that $\text{supp } \widehat{\theta}$ is compact. For $0 < h < h_0$ small enough,

$$\tau_h \frac{H(hk) - H(hj)}{h} \notin \text{supp } \widehat{\theta}, \quad \forall hk, hj \in O \text{ such that } H(hk) \neq H(hj).$$

Therefore, for such h , a and θ ,

$$\begin{aligned} \int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle dt &= \frac{\widehat{\theta}(0)}{(2\pi)^{d/2}} \sum_{\substack{kh, hj \in O \\ H(hk) = H(hj)}} \widehat{u_h}(k) \overline{\widehat{u_h}(j)} \widehat{a}_{j-k} \left(\frac{h}{2}(k+j) \right) \\ &= \widehat{\theta}(0) \sum_{E_h \in H(h\mathbb{Z}^d) \cap H(O)} \langle P_{E_h} u_h, \text{Op}_h(a) P_{E_h} u_h \rangle_{L^2(\mathbb{T}^d)}, \end{aligned}$$

where P_{E_h} stands for the orthogonal projector onto the eigenspace associated to the eigenvalue E_h . This can be rewritten as:

$$\int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle dt = \widehat{\theta}(0) \sum_{E_h \in H(h\mathbb{Z}^d) \cap H(O)} c_h^{E_h} \langle w_{v_h}^h, a \rangle,$$

where

$$v_h^{E_h} := \frac{P_{E_h} u_h}{\|P_{E_h} u_h\|_{L^2(\mathbb{T}^d)}}, \quad \text{and} \quad c_h^{E_h} := \|P_{E_h} u_h\|_{L^2(\mathbb{T}^d)}^2.$$

Note that $v_h^{E_h}$ are eigenfunctions of $H(hD_x)$ and the fact that (u_h) is normalised implies:

$$\sum_{E_h \in H(h\mathbb{Z}^d) \cap H(O)} c_h^{E_h} = 1.$$

We conclude by applying (a straightforward adaptation of) Lemma 6.2 to $v_h^{E_h}$ and $c_h^{E_h}$. \square

Corollary 6.6. *Suppose $\tau_h^H := \tau_h^H(\mathbb{R}^d) \rightarrow \infty$ as $h \rightarrow 0^+$ and that (τ_h) is a time scale such that $\tau_h^H \ll \tau_h$. Then*

$$\widetilde{\mathcal{M}}(\tau) = \text{Conv } \widetilde{\mathcal{M}}(\infty).$$

Proof. The inclusion $\widetilde{\mathcal{M}}(\tau) \subseteq \text{Conv } \widetilde{\mathcal{M}}(\infty)$ is a consequence of the previous result with $O = \mathbb{R}^d$. The converse inclusion can be proved by reversing the steps of the proof of Proposition 6.5. \square

Remark 6.7. *Proposition 1.14 is a direct consequence of this result.*

7. OBSERVABILITY AND UNIQUE CONTINUATION.

In this section we prove Theorem 1.16. Start noticing that the fact that (14) does not hold is equivalent to the existence of a sequence (u_h) in $L^2(\mathbb{T}^d)$ such that:

$$\|\chi(hD_x) u_h\|_{L^2(\mathbb{T}^d)} = 1,$$

and

$$\lim_{h \rightarrow 0^+} \int_0^T \int_U \left| S_h^{t/h} \chi(hD_x) u(x) \right|^2 dx dt = 0.$$

This in turn, is equivalent to the existence of an element $\mu \in \widetilde{\mathcal{M}}(1/h)$ such that:

$$(70) \quad \bar{\mu}(\text{supp } \chi) = 1, \quad \bar{\mu}(C_H) = 0, \quad \int_0^T \mu(t, U \times \text{supp } \chi) dt = 0,$$

(recall that $\bar{\mu}$ is the projection on μ on the ξ -coordinate). This establishes the equivalence between statements (i) and (ii) in Theorem 1.16.

Let $\mu \in \widetilde{\mathcal{M}}(1/h)$ such that $\bar{\mu}(C_H) = 0$. Theorem 1.10 implies that μ decomposes as a sum of positive measures:

$$\mu = \sum_{\Lambda \in \mathcal{L}} \mu_{\Lambda}^{\text{final}},$$

such that, see Remark 4.3 and Theorem 3.2, for any $b \in \mathcal{C}(T^*\mathbb{T}^d)$,

$$\int_{T^*\mathbb{T}^d} b(x, \xi) \mu_{\Lambda}^{\text{final}}(t, dx, d\xi) = \int_{(\langle \Lambda \rangle / \Lambda) \times I_{\Lambda}} \text{Tr} \left(m_{\langle b \rangle_{\Lambda}}(\sigma) N_{\Lambda}(t, \omega, \sigma) \right) \bar{\mu}_{\Lambda}(d\omega, d\sigma),$$

for some $\bar{\mu}_{\Lambda} \in \mathcal{M}_+((\langle \Lambda \rangle / \Lambda) \times \mathbb{R}^d)$ and where $N_{\Lambda}(t, \omega, \sigma)$ is given by:

$$(71) \quad N_{\Lambda}(t, \omega, \sigma) = U_{\Lambda, \omega, \sigma}(t) N_{\Lambda}^0(\omega, \sigma) U_{\Lambda, \omega, \sigma}^*(t),$$

for some positive, self-adjoint trace-class operator $N_\Lambda^0(\omega, \sigma)$ acting on $L_\omega^2(\mathbb{R}^d, \Lambda)$ with $\text{Tr}_{L_\omega^2(\mathbb{R}^d, \Lambda)} N_\Lambda^0(\omega, \sigma) = 1$ and where $U_{\Lambda, \omega, \sigma}(t)$ is the unitary propagator of the equation $(S_{\Lambda, \omega, \sigma})$.

Therefore, the measure $\bar{\mu}_\Lambda$ only charges those $\sigma \in \mathbb{R}^d$ satisfying $\Lambda \subseteq dH(\sigma)^\perp$ (see Remark 4.3) and we also have:

$$\int_{\mathbb{T}^d} \mu_\Lambda^{\text{final}}(dx, \cdot) = \int_{\langle \Lambda \rangle / \Lambda} \bar{\mu}_\Lambda(d\omega, \cdot).$$

If $(\varphi_j^0(\omega, \sigma))_{j \in \mathbb{N}}$ is an orthonormal basis in $L_\omega^2(\mathbb{R}^d, \Lambda)$ consisting of eigenfunctions of the operator $N_\Lambda^0(\omega, \sigma)$ then

$$N_\Lambda^0(\omega, \sigma) = \sum_{j=1}^{\infty} \lambda_j(\omega, \sigma) |\varphi_j^0(\omega, \sigma)\rangle \langle \varphi_j^0(\omega, \sigma)|,$$

where $\sum_{j=1}^{\infty} \lambda_j = 1$ and $\lambda_j \geq 0$. Now (71) implies that:

$$(72) \quad N_\Lambda(t, \omega, \sigma) = \sum_{j=1}^{\infty} \lambda_j(\omega, \sigma) |\varphi_j(t, \omega, \sigma)\rangle \langle \varphi_j(t, \omega, \sigma)|$$

where $\varphi_j(t, \omega, \sigma) \in L_\omega^2(\mathbb{R}^d, \Lambda)$ is the solution to :

$$i\partial_t \varphi_j(t, \omega, \sigma) = \left(\frac{1}{2} d^2 H(\sigma) D_y \cdot D_y + \langle V(t, \sigma) \rangle_\Lambda \right) \varphi_j(t, \omega, \sigma)$$

with $\varphi_j|_{t=0} = \varphi_j^0$.

Now, suppose that Theorem 1.16 (ii) fails. Therefore there exists $\mu \in \widetilde{\mathcal{M}}(1/h)$ which satisfies condition (70). Then there exists $\Lambda \in \mathcal{L}$ such that

$$\mu_\Lambda^{\text{final}}(t, U \times \text{supp } \chi) dt = 0,$$

for every $t \in (0, T)$, but such that $\mu_\Lambda^{\text{final}} \neq 0$. This implies that $\bar{\mu}_\Lambda \neq 0$ and that, for $\bar{\mu}_\Lambda$ -a.e. (ω, σ) with $\Lambda \subseteq \Lambda_\sigma$:

$$(73) \quad \text{Tr}_{L_\omega^2(\mathbb{R}^d, \Lambda)} (\langle \mathbf{1}_U \rangle_\Lambda N_\Lambda(t, \omega, \sigma)) = 0.$$

Comparing with (72), we obtain

$$\int_0^T \int_U |\varphi_j(t, \omega, \sigma)|^2(y) dy dt = 0,$$

for every j such that $\lambda_j \neq 0$ $\bar{\mu}_\Lambda$ -a.e.. Since $\bar{\mu}_\Lambda \neq 0$, $N_\Lambda(\cdot, \omega, \sigma) \neq 0$ on a set of positive $\bar{\mu}_\Lambda$ -measure. This implies that at least for one j , $\varphi_j(\cdot, \omega, \sigma) \neq 0$ and therefore, the unique continuation property of Theorem 1.16 iii) fails for that choice of Λ , ω and σ . This shows that iii) implies ii).

8. APPENDIX: BASIC PROPERTIES OF WIGNER DISTRIBUTIONS AND SEMI-CLASSICAL MEASURES

In this Appendix, we review basic properties of Wigner distributions and semiclassical measures. Recall that we have defined $w_{u_h}^h$ for $u_h \in L^2(\mathbb{T}^d)$ as:

$$(74) \quad \int_{T^*\mathbb{T}^d} a(x, \xi) w_{u_h}^h(dx, d\xi) = \langle u_h, \text{Op}_h(a) u_h \rangle_{L^2(\mathbb{T}^d)}, \quad \text{for all } a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d),$$

Start noticing that (74) admits the more explicit expression:

$$(75) \quad \int_{T^*\mathbb{T}^d} a(x, \xi) w_{u_h}^h(dx, d\xi) = \frac{1}{(2\pi)^{d/2}} \sum_{k, j \in \mathbb{Z}^d} \widehat{u_h}(k) \overline{\widehat{u_h}(j)} \widehat{a}_{j-k} \left(\frac{h}{2}(k+j) \right),$$

where $\widehat{u_h}(k) := \int_{\mathbb{T}^d} u_h(x) \frac{e^{-ik \cdot x}}{(2\pi)^{d/2}} dx$ and $\widehat{a}_k(\xi) := \int_{\mathbb{T}^d} a(x, \xi) \frac{e^{-ik \cdot x}}{(2\pi)^{d/2}} dx$ denote the respective Fourier coefficients of u_h and a , with respect to the variable $x \in \mathbb{T}^d$.

By the Calderón-Vaillancourt theorem [11], the norm of $\text{Op}_h(a)$ is uniformly bounded in h : indeed, there exists an integer K_d , and a constant $C_d > 0$ (depending on the dimension d) such that, if a is a smooth function on $T^*\mathbb{T}^d$, with uniformly bounded derivatives, then

$$\|\text{Op}_1(a)\|_{\mathcal{L}(L^2(\mathbb{T}^d))} \leq C_d \sum_{\alpha \in \mathbb{N}^{2d}, |\alpha| \leq K_d} \sup_{T^*\mathbb{T}^d} |\partial^\alpha a| =: C_d M(a).$$

A proof in the case of $L^2(\mathbb{R}^d)$ can be found in [16]. As a consequence of this, equation (74) gives:

$$\left| \int_{T^*\mathbb{T}^d} a(x, \xi) w_{u_h}^h(dx, d\xi) \right| \leq C_d \|u_h\|_{L^2(\mathbb{T}^d)}^2 M(a), \quad \text{for all } a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d).$$

Therefore, if $w_h(t, \cdot) := w_{S_h^{\tau_h t} u_h}^h$ for some function $h \mapsto \tau_h \in \mathbb{R}_+$ and if the family (u_h) is bounded in $L^2(\mathbb{T}^d)$ one has that (w_h) is uniformly bounded in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$. Let us consider $\mu \in L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$ an accumulation point of (w_h) for the weak-* topology.

It follows from standard results on the Weyl quantization that μ enjoys the following properties :

- (a) $\mu \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$, meaning that for almost all t , $\mu(t, \cdot)$ is a positive measure on $T^*\mathbb{T}^d$.
- (b) The unitary character of S_h^t implies that $\int_{T^*\mathbb{T}^d} \mu(t, dx, d\xi)$ does not depend on t ; from the normalization of u_h , we have $\int_{T^*\mathbb{T}^d} \mu(\tau, dx, d\xi) \leq 1$, the inequality coming from the fact that $T^*\mathbb{T}^d$ is not compact, and that there may be an escape of mass to infinity. Such escape does not occur if and only if (u_h) is h -oscillating, in which case $\mu \in L^\infty(\mathbb{R}; \mathcal{P}(T^*\mathbb{T}^d))$.
- (c) If $\tau_h \rightarrow \infty$ as $h \rightarrow 0^+$ then the measures $\mu(t, \cdot)$ are invariant under ϕ_s , for almost all t and all s .

- (d) Let $\bar{\mu}$ be the measure on \mathbb{R}^d given by the image of $\mu(t, \cdot)$ under the projection map $(x, \xi) \mapsto \xi$. If $\mathbf{V}_h(t) = \text{Op}_h(V(t, x, \xi))$ is a pseudodifferential operator and if $\tau_h \ll h^{-2}$ then $\bar{\mu}$ does not depend on t . Moreover, if $\overline{\mu_0}$ stands for the image under the same projection of any semiclassical measure corresponding to the sequence of initial data (u_h) then $\bar{\mu} = \overline{\mu_0}$.

For the reader's convenience, we next prove statements (c) and (d) (see also [28] for a proof of these results in the context of the Schrödinger flow $e^{iht\Delta}$ on a general Riemannian manifold). Let us begin with the invariance through the Hamiltonian flow. We set

$$a_s(x, \xi) := a(x + sdH(\xi), \xi) = a \circ \phi_s(x, \xi).$$

The symbolic calculus for Weyl's quantization implies:

$$\frac{d}{ds} S_h^s \text{Op}_h(a_s) S_h^{s*} = S_h^s \text{Op}_h(\partial_s a_s) S_h^{s*} - \frac{i}{h} S_h^s [H(hD) + h^2 \mathbf{V}_h(t), \text{Op}_h(a_s)] S_h^{s*} = O(h).$$

Therefore, for fixed s , $S_h^s \text{Op}_h(a_s) S_h^{s*} = \text{Op}_h(a) + O(h)$ (note that we have only used here the boundedness of the operator $\mathbf{V}_h(t)$) and for $\theta \in L^1(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle dt &= \int_{\mathbb{R}} \theta(t) \langle u_h, S_h^{\tau_h t*} \text{Op}_h(a) S_h^{\tau_h t} u_h \rangle dt \\ &= \int_{\mathbb{R}} \theta(t) \langle u_h, S_h^{\tau_h(t-s/\tau_h)*} \text{Op}_h(a \circ \phi_s) S_h^{\tau_h(t-s/\tau_h)} u_h \rangle dt + O(h) \\ &= \int_{\mathbb{R}} \theta(t + s/\tau_h) \langle u_h, S_h^{\tau_h t*} \text{Op}_h(a \circ \phi_s) S_h^{\tau_h t} u_h \rangle dt + O(h) \\ &= \int_{\mathbb{R}} \theta(t + s/\tau_h) \langle w_h(t), a \circ \phi_s \rangle dt + O(h). \end{aligned}$$

Since $\|\theta(\cdot + s/\tau_h) - \theta\|_{L^1} \rightarrow 0$ (recall that we have assumed that $\tau_h \rightarrow \infty$ as $h \rightarrow 0^+$) we obtain

$$\int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle dt - \int_{\mathbb{R}} \theta(t) \langle w_h(t), a \circ \phi_s \rangle dt \rightarrow 0, \text{ as } h \rightarrow 0^+,$$

whence the invariance under ϕ_s .

Let us now prove property (d). Consider $\bar{\mu}$ the image of μ by the projection $(x, \xi) \mapsto \xi$, we have for $a \in \mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned} \langle w_h(t), a(\xi) \rangle - \langle w_{u_h}^h, a(\xi) \rangle &= \int_0^t \frac{d}{ds} \langle w_h(s), a(\xi) \rangle ds \\ &= \int_0^t \langle u_h, \frac{d}{ds} (S_h^{\tau_h s*} \text{Op}_h(a) S_h^{\tau_h s}) u_h \rangle ds \\ &= O(\tau_h h \|[\mathbf{V}_h(t), \text{Op}_h(a)]\|_{\mathcal{L}(L^2(\mathbb{T}^d))}), \end{aligned}$$

(for a only depending on ξ we have $\text{Op}_h(a) = a(hD_x)$, which commutes with $H(hD_x)$). If $\mathbf{V}_h(t) = \text{Op}_h(V(t, x, \xi))$ then

$$\|[\mathbf{V}_h(t), \text{Op}_h(a)]\|_{\mathcal{L}(L^2(\mathbb{T}^d))} = O(h) \text{ and } \tau_h h \|[\mathbf{V}_h(t), \text{Op}_h(a)]\|_{\mathcal{L}(L^2(\mathbb{T}^d))} = O(\tau_h h^2).$$

Therefore, if $\tau_h \ll h^{-2}$, we find, for every $\theta \in L^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \theta(t) \int_{T^*\mathbb{T}^d} a(\xi) \mu(t, dx, d\xi) = \left(\int_{\mathbb{R}} \theta(t) dt \right) \int_{T^*\mathbb{T}^d} a(\xi) \mu_0(dx, d\xi),$$

where μ_0 is any accumulation point of $(w_{u_h}^h)$. As a consequence of this, we find that $\bar{\mu}$ does not depend on t and:

$$\bar{\mu}(\xi) = \int_{\mathbb{T}^d} \mu_0(dy, \xi).$$

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